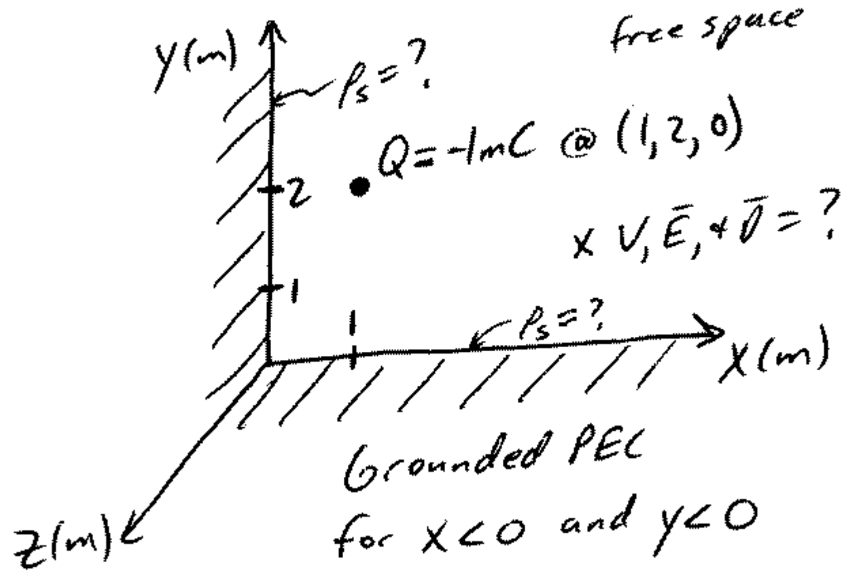
ex. For the problem shown below, find

$V, \bar{E}, \bar{D}, + \rho_s$.



→ Extremely difficult problem to attempt to solve directly as there is a mix of a known point charge Q, unknown surface charge density ρ_s, and 2 known boundary conditions ($V=0$ @ $x=0$ and $V=0$ @ $y=0$).

$$\Downarrow$$

Use method of Images to create an equivalent problem

ex. cont.

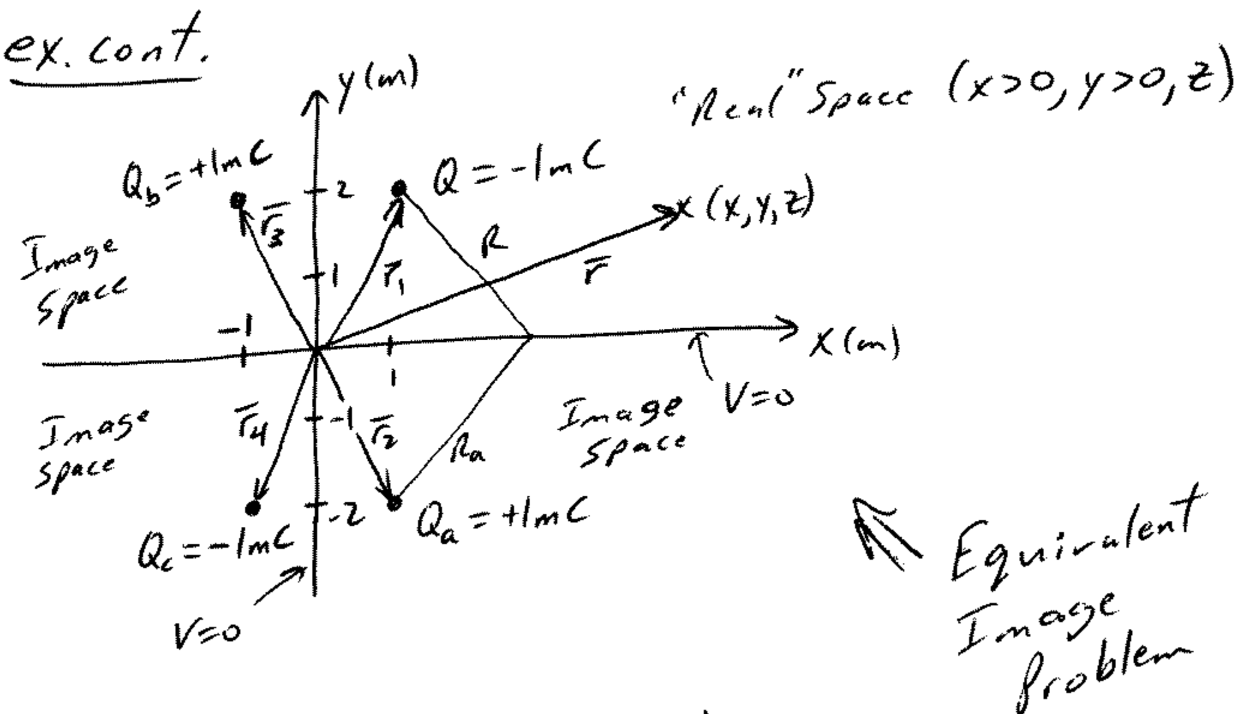


Image Charges (Q_a, Q_b, + Q_c)

1) Need to maintain $V=0$ on the plane $y=0$

$\Rightarrow$ place image charge $Q_a = -Q$ @ $(1,-2,0)$, then

on the plane $y=0$ $\quad V_Q + V_{Q_a} = \dfrac{Q}{4\pi\epsilon_0 R} + \dfrac{Q_a}{4\pi\epsilon_0 R_a}$

$$= \frac{Q}{4\pi\epsilon_0 R} + \frac{-Q}{4\pi\epsilon_0 R} = \underline{\underline{0}}$$

2) Similarly, to maintain $V=0$ on the plane $x=0$,

place image charges $Q_b = -Q$ and $Q_c = -Q_a = Q$ at $(-1,2,0)$ and $(-1,-2,0)$ respectively.

Note that Q_b + Q_c also maintain $V=0$ on the plane $y=0$.

ex. cont.

With the equivalent image problem defined, find V using $V = \sum_{k=1}^{4} \frac{Q_k}{4\pi\epsilon_0 |\bar{r} - \bar{r}_k|}$

where $\bar{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ (field position vector)

$$\bar{r}_1 = \bar{r}_Q = \hat{a}_x + 2\hat{a}_y \ (m) \qquad Q_1 = Q = -10^{-3}\,C$$

$$\bar{r}_2 = \bar{r}_{Q_a} = \hat{a}_x - 2\hat{a}_y \ (m) \qquad Q_2 = Q_a = +10^{-3}\,C$$

$$\bar{r}_3 = \bar{r}_{Q_b} = -\hat{a}_x + 2\hat{a}_y \ (m) \qquad Q_3 = Q_b = +10^{-3}\,C$$

$$\bar{r}_4 = \bar{r}_{Q_c} = -\hat{a}_x - 2\hat{a}_y \ (m) \qquad Q_4 = Q_c = -10^{-3}\,C$$

$$V(x,y,z) = \frac{-10^{-3}}{4\pi\epsilon_0 |(x-1)\hat{a}_x + (y-2)\hat{a}_y + z\hat{a}_z|} + \frac{10^{-3}}{4\pi\epsilon_0 |(x-1)\hat{a}_x + (y+2)\hat{a}_y + z\hat{a}_z|}$$

$$+ \frac{10^{-3}}{4\pi\epsilon_0 |(x+1)\hat{a}_x + (y-2)\hat{a}_y + z\hat{a}_z|} + \frac{-10^{-3}}{4\pi\epsilon_0 |(x+1)\hat{a}_x + (y+2)\hat{a}_y + z\hat{a}_z|}$$

$$\boxed{\begin{aligned} V(x,y,z) = 8.9877 \Bigg[&\frac{-1}{\sqrt{(x-1)^2 + (y-2)^2 + z^2}} + \frac{1}{\sqrt{(x-1)^2 + (y+2)^2 + z^2}} \\ &+ \frac{1}{\sqrt{(x+1)^2 + (y-2)^2 + z^2}} + \frac{-1}{\sqrt{(x+1)^2 + (y+2)^2 + z^2}} \Bigg] \text{ MV} \end{aligned}}$$

valid for $x > 0, y > 0$, + any z

$V(x,y,z) = 0$ elsewhere (w/in PEC)

ex. cont.

Next, find $\bar{E}$ using $\bar{E} = -\bar{\nabla} V$ or $\sum_{k=1}^{4} \frac{Q_k(\bar{r}-\bar{r}_k)}{4\pi\epsilon_0 |\bar{r}-\bar{r}_k|^3}$.
I'll use the latter equation as all the necessary info is already available from the voltage calculations.

$$\bar{E}(x,y,z) = 8.9877\left[\frac{-(x-1)\hat{a}_x-(y-2)\hat{a}_y-z\hat{a}_z}{\left[(x-1)^2+(y-2)^2+z^2\right]^{3/2}} + \frac{(x-1)\hat{a}_x+(y+2)\hat{a}_y+z\hat{a}_z}{\left[(x-1)^2+(y+2)^2+z^2\right]^{3/2}} + \frac{(x+1)\hat{a}_x+(y-2)\hat{a}_y+z\hat{a}_z}{\left[(x+1)^2+(y-2)^2+z^2\right]^{3/2}} + \frac{-(x+1)\hat{a}_x-(y+2)\hat{a}_y-z\hat{a}_z}{\left[(x+1)^2+(y+2)^2+z^2\right]^{3/2}}\right] \frac{mV}{m}$$

valid for $x>0$, $y>0$, & any z

$\bar{E}(x,y,z) = 0$ elsewhere

$$\bar{D} = \epsilon_0 \bar{E} = 79.58\left[\frac{-(x-1)\hat{a}_x-(y-2)\hat{a}_y-z\hat{a}_z}{\left[(x-1)^2+(y-2)^2+z^2\right]^{3/2}} + \frac{(x-1)\hat{a}_x+(y+2)\hat{a}_y+z\hat{a}_z}{\left[(x-1)^2+(y+2)^2+z^2\right]^{3/2}} + \frac{(x+1)\hat{a}_x+(y-2)\hat{a}_y+z\hat{a}_z}{\left[(x+1)^2+(y-2)^2+z^2\right]^{3/2}} + \frac{-(x+1)\hat{a}_x-(y+2)\hat{a}_y-z\hat{a}_z}{\left[(x+1)^2+(y+2)^2+z^2\right]^{3/2}}\right] \mu C/m^2$$

$x>0$, $y>0$, any z

$\bar{D} = 0$ elsewhere

To find ρ_s on the conducting surfaces, use the boundary condition $\rho_s = D_{normal}$.

ex. cont.

Vertical surface

$$\rho_s(x=0, y>0, z) = \hat{a}_x \cdot \overline{D}(x=0, y>0, z)$$

$$= 79.58\left[\frac{1}{[1^2+(y-2)^2+z^2]^{3/2}} + \frac{-1}{[1^2+(y+2)^2+z^2]^{3/2}} + \frac{1}{[1^2+(y-2)^2+z^2]^{3/2}} + \frac{-1}{[1^2+(y+2)^2+z^2]^{3/2}}\right]$$

$$\rho_s(x=0, y>0, z) = 159.15\left[\frac{1}{[1+(y-2)^2+z^2]^{3/2}} - \frac{1}{[1+(y+2)^2+z^2]^{3/2}}\right]\frac{\mu C}{m^2}$$

Horizontal Surface

$$\rho_s(x>0, y=0, z) = \hat{a}_y \cdot \overline{D}(x>0, y=0, z)$$

$$= 79.58\left[\frac{2}{[(x-1)^2+2^2+z^2]^{3/2}} + \frac{2}{[(x-1)^2+2^2+z^2]^{3/2}} + \frac{-2}{[(x+1)^2+2^2+z^2]^{3/2}} + \frac{-2}{[(x+1)^2+2^2+z^2]^{3/2}}\right]$$

$$\rho_s(x>0, y=0, z) = 318.31\left[\frac{1}{[(x-1)^2+4+z^2]^{3/2}} - \frac{1}{[(x+1)^2+4+z^2]^{3/2}}\right]\mu C/m^2$$

$\rho_s = 0$ elsewhere

<u>Plot surface charge density versus position</u> [Q = -1mC at (x = 1, y = 2, z = 0)]

$n := 1..601 \qquad x_n := (n-1)\cdot 0.01 \qquad y_n := (n-1)\cdot 0.01 \qquad z := 0$

$$\rho s\_vert_n := 159.5\cdot\left[\frac{1}{\left[1+(y_n-2)^2+z^2\right]^{1.5}}-\frac{1}{\left[1+(y_n+2)^2+z^2\right]^{1.5}}\right]$$

$$\rho s\_horz_n := 318.31\cdot\left[\frac{1}{\left[(x_n-1)^2+4+z^2\right]^{1.5}}-\frac{1}{\left[(x_n+1)^2+4+z^2\right]^{1.5}}\right]$$