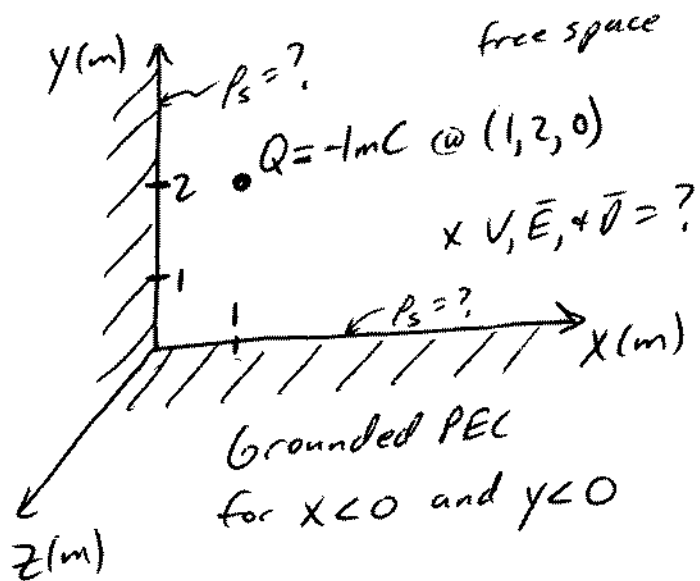


EX. For the problem shown below, find  $V, \vec{E}, \vec{D}, + \rho_s$ .

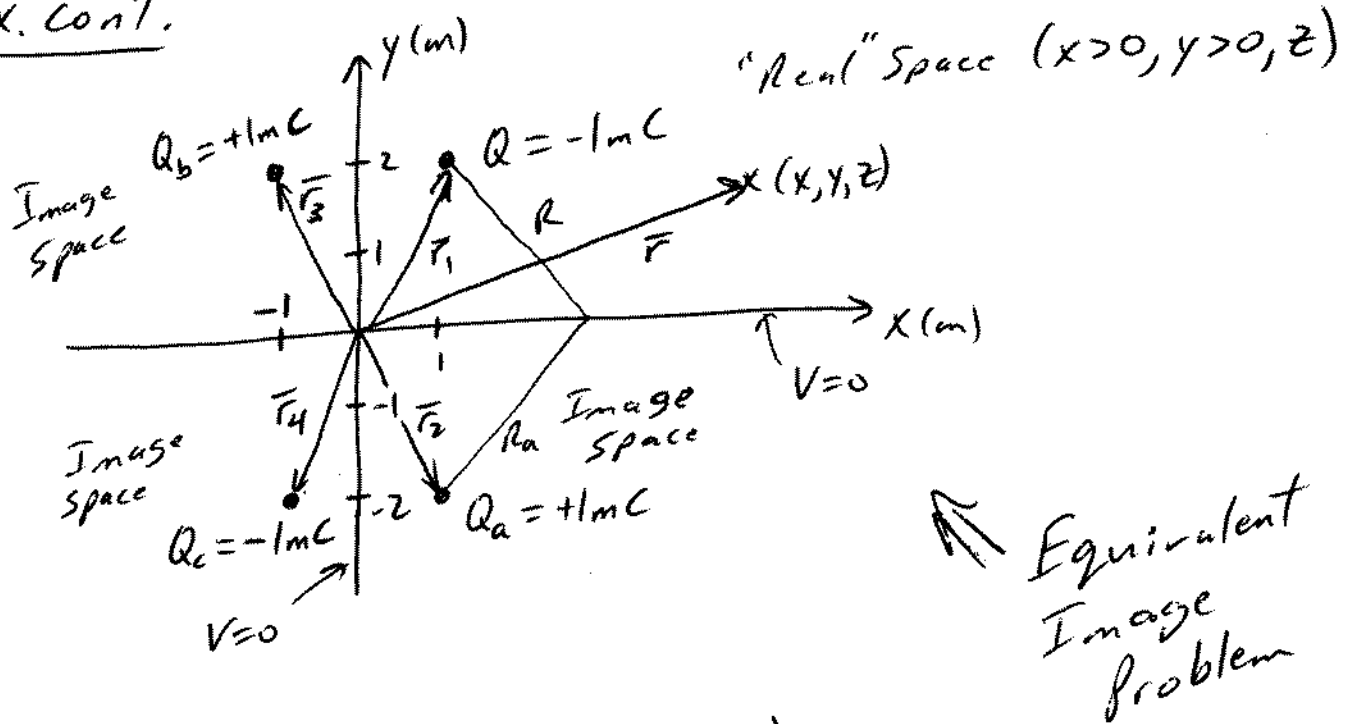


→ Extremely difficult problem to attempt to solve directly as there is a mix of a known point charge  $Q$ , unknown surface charge density  $\rho_s$ , and 2 known boundary conditions ( $V=0 @ x=0$  and  $V=0 @ y=0$ ).



Use method of Images to create an equivalent problem

ex. cont.



### Image Charges ( $Q_a, Q_b, + Q_c$ )

1) Need to maintain  $V=0$  on the plane  $y=0$

$\Rightarrow$  place image charge  $Q_a = -Q$  @  $(1, -2, 0)$ , then

$$\begin{aligned} \text{on the plane } y=0 \quad V_Q + V_{Q_a} &= \frac{Q}{4\pi\epsilon_0 R} + \frac{Q_a}{4\pi\epsilon_0 R_a} \\ &= \frac{Q}{4\pi\epsilon_0 R} + \frac{-Q}{4\pi\epsilon_0 R} = \underline{\underline{0}} \end{aligned}$$

2) Similarly, to maintain  $V=0$  on the plane  $x=0$ ,

place image charges  $Q_b = -Q$  and  $Q_c = -Q_a = Q$  at  $(-1, 2, 0)$  and  $(-1, -2, 0)$  respectively.

Note that  $Q_b + Q_c$  also maintain  $V=0$  on the plane  $y=0$ .

Ex. cont.

With the equivalent image problem defined, find

$$V, \text{ using } V = \sum_{k=1}^4 \frac{Q_k}{4\pi\epsilon_0 |\vec{r} - \vec{r}_k|}$$

where  $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$  (field position vector)

$$\vec{r}_1 = \vec{r}_Q = \hat{a}_x + 2\hat{a}_y \text{ (m)} \quad Q_1 = Q = -10^{-3} \text{ C}$$

$$\vec{r}_2 = \vec{r}_{Q_a} = \hat{a}_x - 2\hat{a}_y \text{ (m)} \quad Q_2 = Q_a = +10^{-3} \text{ C}$$

$$\vec{r}_3 = \vec{r}_{Q_b} = -\hat{a}_x + 2\hat{a}_y \text{ (m)} \quad Q_3 = Q_b = +10^{-3} \text{ C}$$

$$\vec{r}_4 = \vec{r}_{Q_c} = -\hat{a}_x - 2\hat{a}_y \text{ (m)} \quad Q_4 = Q_c = -10^{-3} \text{ C}$$

$$V(x, y, z) = \frac{-10^{-3}}{4\pi\epsilon_0 |(x-1)\hat{a}_x + (y-2)\hat{a}_y + z\hat{a}_z|} + \frac{10^{-3}}{4\pi\epsilon_0 |(x-1)\hat{a}_x + (y+2)\hat{a}_y + z\hat{a}_z|} \\ + \frac{10^{-3}}{4\pi\epsilon_0 |(x+1)\hat{a}_x + (y-2)\hat{a}_y + z\hat{a}_z|} + \frac{-10^{-3}}{4\pi\epsilon_0 |(x+1)\hat{a}_x + (y+2)\hat{a}_y + z\hat{a}_z|}$$

$$V(x, y, z) = 8.9877 \left[ \frac{-1}{\sqrt{(x-1)^2 + (y-2)^2 + z^2}} + \frac{1}{\sqrt{(x-1)^2 + (y+2)^2 + z^2}} \right. \\ \left. + \frac{1}{\sqrt{(x+1)^2 + (y-2)^2 + z^2}} + \frac{-1}{\sqrt{(x+1)^2 + (y+2)^2 + z^2}} \right] \text{ MV}$$

valid for  $x > 0, y > 0, + \text{ any } z$

$V(x, y, z) = 0$  elsewhere (w/in PEC)

ex. cont.

Next, find  $\vec{E}$  using  $\vec{E} = -\nabla V$  or  $\sum_{k=1}^4 \frac{Q_k (\vec{r} - \vec{r}_k)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_k|^3}$ .

I'll use the latter equation as all the necessary info is already available from the voltage calculations.

$$\vec{E}(x, y, z) = 8.9877 \left[ \frac{-(x-1)\hat{a}_x - (y-2)\hat{a}_y - z\hat{a}_z}{[(x-1)^2 + (y-2)^2 + z^2]^{3/2}} + \frac{(x-1)\hat{a}_x + (y+2)\hat{a}_y + z\hat{a}_z}{[(x-1)^2 + (y+2)^2 + z^2]^{3/2}} \right. \\ \left. + \frac{(x+1)\hat{a}_x + (y-2)\hat{a}_y + z\hat{a}_z}{[(x+1)^2 + (y-2)^2 + z^2]^{3/2}} + \frac{-(x+1)\hat{a}_x - (y+2)\hat{a}_y - z\hat{a}_z}{[(x+1)^2 + (y+2)^2 + z^2]^{3/2}} \right] \frac{mV}{m}$$

valid for  $x > 0, y > 0, \forall$  any  $z$

$$\vec{E}(x, y, z) = 0 \quad \text{elsewhere}$$

$$\vec{D} = \epsilon_0 \vec{E} = \begin{cases} 79.58 [\text{ditto}] \frac{\mu C}{m^2} & x > 0, y > 0, \text{ any } z \\ 0 & \text{elsewhere} \end{cases}$$

To find  $\rho_s$  on the conducting surfaces, use the boundary condition  $\rho_s = D_{\text{normal}}$ .

ex. cont.

Vertical surface

$$P_s(x=0, y>0, z) = \hat{a}_x \cdot \bar{D}(x=0, y>0, z)$$

$$= 79.58 \left[ \frac{1}{[1^2 + (y-2)^2 + z^2]^{3/2}} + \frac{-1}{[1^2 + (y+2)^2 + z^2]^{3/2}} \right. \\ \left. + \frac{1}{[1^2 + (y-2)^2 + z^2]^{3/2}} + \frac{-1}{[1^2 + (y+2)^2 + z^2]^{3/2}} \right]$$

$$P_s(x=0, y>0, z) = 159.15 \left[ \frac{1}{[1 + (y-2)^2 + z^2]^{3/2}} - \frac{1}{[1 + (y+2)^2 + z^2]^{3/2}} \right] \frac{\mu C}{m^2}$$

Horizontal surface

$$P_s(x>0, y=0, z) = \hat{a}_y \cdot \bar{D}(x>0, y=0, z)$$

$$= 79.58 \left[ \frac{2}{[(x-1)^2 + 2^2 + z^2]^{3/2}} + \frac{2}{[(x-1)^2 + 2^2 + z^2]^{3/2}} \right. \\ \left. + \frac{-2}{[(x+1)^2 + 2^2 + z^2]^{3/2}} + \frac{-2}{[(x+1)^2 + 2^2 + z^2]^{3/2}} \right]$$

$$P_s(x>0, y=0, z) = 318.31 \left[ \frac{1}{[(x-1)^2 + 4 + z^2]^{3/2}} - \frac{1}{[(x+1)^2 + 4 + z^2]^{3/2}} \right] \frac{\mu C}{m^2}$$

$P_s = 0$  elsewhere

Plot surface charge density versus position [ $Q = -1\text{mC}$  at  $(x = 1, y = 2, z = 0)$ ]

$n := 1 .. 601$      $x_n := (n - 1) \cdot 0.01$      $y_n := (n - 1) \cdot 0.01$      $z := 0$

$$\rho_{s\_vert_n} := 159.5 \cdot \left[ \frac{1}{\left[1 + (y_n - 2)^2 + z^2\right]^{1.5}} - \frac{1}{\left[1 + (y_n + 2)^2 + z^2\right]^{1.5}} \right]$$

$$\rho_{s\_horz_n} := 318.31 \cdot \left[ \frac{1}{\left[(x_n - 1)^2 + 4 + z^2\right]^{1.5}} - \frac{1}{\left[(x_n + 1)^2 + 4 + z^2\right]^{1.5}} \right]$$

