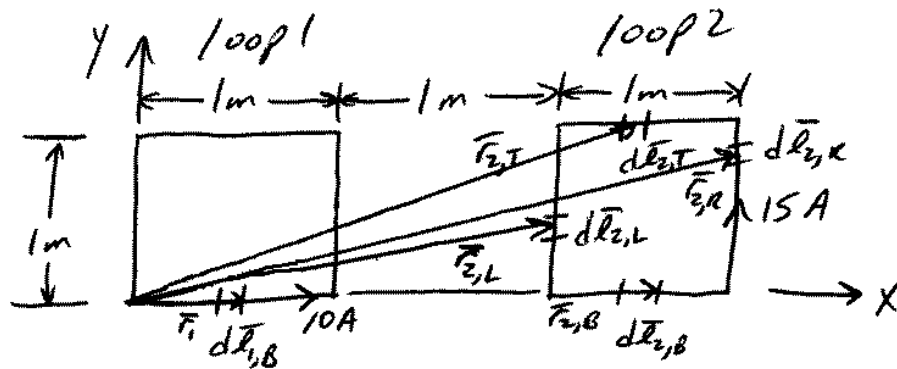


EX. For the two square loops shown, find the force exerted on the bottom of loop 1 due to the magnetic field generated by the current flowing around loop 2.



The appropriate force equation is

$$\vec{F}_{1,Bott} = \frac{\mu_0 I_1 I_2}{4\pi} \int_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \times [d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)]}{|\vec{r}_1 - \vec{r}_2|^3}$$

(Bottom)

$$\vec{F}_{1,B} = \vec{F}_{1,Bott} + \vec{F}_{1,Bott} + \vec{F}_{1,Bott} + \vec{F}_{1,Bott}$$

2, Bott      2, Top      2, Left      2, Right

Contour  $C_2$  for  
loop 2 has  
4 sides

Find/define pieces of integrals

$$I_1 = 10A, \quad I_2 = 15A, \quad d\vec{l}_{1,B} = dx_1 \hat{a}_x \quad (0 < x_1 < 1m), \quad \vec{r}_1 = x_1 \hat{a}_x$$

$$d\vec{l}_{2,B} = dx_{2B} \hat{a}_x \quad (2 < x_{2B} < 3m), \quad \vec{r}_{2,B} = x_{2B} \hat{a}_x$$

$$d\vec{l}_{2,T} = dx_{2T} \hat{a}_x \quad (2 < x_{2T} < 3m), \quad \vec{r}_{2,T} = x_{2T} \hat{a}_x + 1 \hat{a}_y \quad (m)$$

$$d\vec{l}_{2,L} = dy_{2L} \hat{a}_y \quad (0 < y_{2L} < 1m), \quad \vec{r}_{2,L} = 2 \hat{a}_x + y_{2L} \hat{a}_y \quad (m)$$

$$d\vec{l}_{2,R} = dy_{2R} \hat{a}_y \quad (0 < y_{2R} < 1m), \quad \vec{r}_{2,R} = 3 \hat{a}_x + y_{2R} \hat{a}_y \quad (m)$$

ex. cont.

$$\vec{F}_{1,B} = \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^1 \int_{x_{2B}=2}^3 \frac{dx_1 \hat{a}_x \times [dx_{2B} \hat{a}_x \times (x_1 \hat{a}_x - x_{2B} \hat{a}_x)]}{|x_1 \hat{a}_x - x_{2B} \hat{a}_x|^3}$$

$\rightarrow 0 \quad (\hat{a}_x \times \hat{a}_x = 0)$

$$+ \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^1 \int_{x_{2T}=3}^2 \frac{dx_1 \hat{a}_x \times [dx_{2T} \hat{a}_x \times (x_1 \hat{a}_x - x_{2T} \hat{a}_x - \hat{a}_y)]}{|x_1 \hat{a}_x - x_{2T} \hat{a}_x - \hat{a}_y|^3}$$

$$+ \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^1 \int_{y_{2L}=1}^0 \frac{dx_1 \hat{a}_x \times [dy_{2L} \hat{a}_y \times (x_1 \hat{a}_x - 2\hat{a}_x - y_{2L} \hat{a}_y)]}{|x_1 \hat{a}_x - 2\hat{a}_x - y_{2L} \hat{a}_y|^3}$$

$$+ \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^1 \int_{y_{2R}=0}^1 \frac{dx_1 \hat{a}_x \times [dy_{2R} \hat{a}_y \times (x_1 \hat{a}_x - 3\hat{a}_x - y_{2R} \hat{a}_y)]}{|x_1 \hat{a}_x - 3\hat{a}_x - y_{2R} \hat{a}_y|^3}$$

$$\vec{F}_{1B} = 0 + 1.5 \times 10^{-5} \int_{x_1=0}^1 \int_{x_{2T}=3}^2 \frac{dx_1 \hat{a}_x \times [0 - 0 - dx_{2T} \hat{a}_z]}{[(x_1 - x_{2T})^2 + (-1)^2]^{3/2}}$$

$$+ 1.5 \times 10^{-5} \int_{x_1=0}^1 \int_{y_{2L}=1}^0 \frac{dx_1 \hat{a}_x \times [-x_1 dy_{2L} \hat{a}_z + 2 dy_{2L} \hat{a}_z + 0]}{[(x_1 - 2)^2 + (-y_{2L})^2]^{3/2}}$$

$$+ 1.5 \times 10^{-5} \int_{x_1=0}^1 \int_{y_{2R}=0}^1 \frac{dx_1 \hat{a}_x \times [-x_1 dy_{2R} \hat{a}_z + 3 dy_{2R} \hat{a}_z - 0]}{[(x_1 - 3)^2 + (-y_{2R})^2]^{3/2}}$$

ex. cont.

$$\vec{F}_{1B} = 1.5 \times 10^{-5} \int_{x_1=0}^1 \int_{x_{2T}=3}^2 \frac{+\hat{a}_y dx_{2T} dx_1}{[(x_1 - x_{2T})^2 + 1^2]^{3/2}}$$

$$+ 1.5 \times 10^{-5} \int_{x_1=0}^1 \int_{y_{2L}=1}^0 \frac{+\hat{a}_y (x_1 - 2) dy_{2L} dx_1}{[(x_1 - 2)^2 + y_{2L}^2]^{3/2}}$$

$$+ 1.5 \times 10^{-5} \int_{x_1=0}^1 \int_{y_{2R}=0}^1 \frac{+\hat{a}_y (x_1 - 3) dy_{2R} dx_1}{[(x_1 - 3)^2 + y_{2R}^2]^{3/2}}$$

Method 1 - use MathCad to Solve integrals

$$\int_0^1 \int_3^2 \frac{1}{[(x_1 - x_{2T})^2 + 1^2]^{1.5}} dx_{2T} dx_1 = -0.1043553$$

$$\int_0^1 \int_1^0 \frac{(x_1 - 2)}{[(x_1 - 2)^2 + y_{2L}^2]^{1.5}} dy_{2L} dx_1 = 0.4001618$$

$$\int_0^1 \int_0^1 \frac{(x_1 - 3)}{[(x_1 - 3)^2 + y_{2R}^2]^{1.5}} dy_{2R} dx_1 = -0.1537617$$

ex. cont.Method 1 cont.

$$\vec{F}_{1B} = \hat{a}_y 1.5 \times 10^{-5} [-0.1043553 + 0.4001618 - 0.1537617]$$

$$\vec{F}_{1B} = \hat{a}_y 2.1307 \mu\text{N}$$

Method 2 - solve integrals analytically

use  $\int \frac{dx}{[(a-x)^2 + b^2]^{3/2}} = \frac{x-a}{b^2 \sqrt{(a-x)^2 + b^2}}$  ← 1<sup>st</sup> term wrt  $x_2$

and

$\int \frac{(x-a) dx}{[(x-a)^2 + b^2]^{3/2}} = \frac{-1}{\sqrt{(x-a)^2 + b^2}}$  ← 2<sup>nd</sup> + 3<sup>rd</sup> terms wrt  $x_1$

$$\vec{F}_{1B} = \hat{a}_y 1.5 \times 10^{-5} \int_{x_1=0}^1 \left( \frac{x_{2T} - x_1}{1^2 \sqrt{(x_1 - x_{2T})^2 + 1^2}} \right) \Big|_{x_{2T}=3}^2 dx_1$$

$$+ \hat{a}_y 1.5 \times 10^{-5} \int_{y_{2L}=1}^0 \left( \frac{-1}{\sqrt{(x_1 - 2)^2 + y_{2L}^2}} \right) \Big|_{x_1=0}^1 dy_{2L}$$

$$+ \hat{a}_y 1.5 \times 10^{-5} \int_{y_{2L}=0}^1 \left( \frac{-1}{\sqrt{(x_1 - 3)^2 + y_{2L}^2}} \right) \Big|_{x_1=0}^1 dy_{2L}$$

ex. cont.Method 2 cont.

$$\bar{F}_{18} = \hat{a}_y 1.5 \times 10^{-5} \int_{x_1=0}^1 \left[ \frac{(2-x_1)}{\sqrt{(x_1-2)^2+1^2}} - \frac{3-x_1}{\sqrt{(x_1-3)^2+1^2}} \right] dx_1$$

$$+ \hat{a}_y 1.5 \times 10^{-5} \int_{y_{2L}=1}^0 \left[ \frac{-1}{\sqrt{y_{2L}^2+1^2}} - \frac{-1}{\sqrt{y_{2L}^2+2^2}} \right] dy_{2L}$$

$$+ \hat{a}_y 1.5 \times 10^{-5} \int_{y_{2R}=0}^1 \left[ \frac{-1}{\sqrt{y_{2R}^2+2^2}} - \frac{-1}{\sqrt{y_{2R}^2+3^2}} \right] dy_{2R}$$

Now use  $\int \frac{(x-a)}{\sqrt{(x-a)^2+b^2}} dx = \sqrt{(x-a)^2+b^2}$

and  $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2})$

$$\begin{aligned} \bar{F}_{18} = & \hat{a}_y 1.5 \times 10^{-5} \left[ -\sqrt{(x_1-2)^2+1^2} + \sqrt{(x_1-3)^2+1^2} \right] \Big|_{x_1=0}^1 \\ & + \hat{a}_y 1.5 \times 10^{-5} \left[ -\ln(y_{2L} + \sqrt{y_{2L}^2+1^2}) + \ln(y_{2L} + \sqrt{y_{2L}^2+2^2}) \right] \Big|_{y_{2L}=1}^0 \\ & + \hat{a}_y 1.5 \times 10^{-5} \left[ -\ln(y_{2R} + \sqrt{y_{2R}^2+2^2}) + \ln(y_{2R} + \sqrt{y_{2R}^2+3^2}) \right] \Big|_{y_{2R}=0}^1 \end{aligned}$$

ex. cont.

Method 2 cont.

$$\begin{aligned} \vec{F}_{1B} = & \hat{a}_y 1.5 \times 10^{-5} \left[ \left( -\sqrt{(1-\sqrt{2})^2 + 1^2} + \sqrt{(0-\sqrt{2})^2 + 1^2} \right) + \left( \sqrt{(1-\sqrt{3})^2 + 1^2} - \sqrt{(0-\sqrt{3})^2 + 1^2} \right) \right] \\ & + \hat{a}_y 1.5 \times 10^{-5} \left[ \left( -\ln(0/\sqrt{0+1}) + \ln(1+\sqrt{1^2+1^2}) \right) + \left( \ln(0/\sqrt{0+2^2}) - \ln(1+\sqrt{1^2+2^2}) \right) \right] \\ & + \hat{a}_y 1.5 \times 10^{-5} \left[ \left( -\ln(1+\sqrt{1^2+2^2}) + \ln(0/\sqrt{0+2^2}) \right) + \left( \ln(1+\sqrt{1^2+3^2}) - \ln(0/\sqrt{0+3^2}) \right) \right] \end{aligned}$$

$$= \hat{a}_y 1.5 \times 10^{-5} (-0.1043553)$$

$$+ \hat{a}_y 1.5 \times 10^{-5} (0.40016176)$$

$$+ \hat{a}_y 1.5 \times 10^{-5} (-0.1537617)$$

Same integral results as calculated by MathCad, just lots more work

$$\underline{\underline{\vec{F}_{1B} = \hat{a}_y 2.1307 \mu\text{N}}} \quad \leftarrow \text{Same answer}$$

⇒ To find complete force on loop 1 would require repeating this process for the other 3 sides!