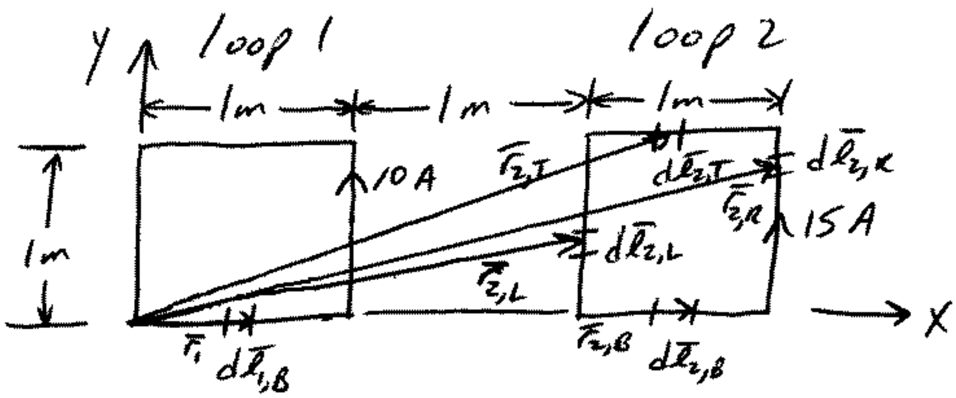
ex. For the two square loops shown, find the force exerted on the bottom of loop 1 due to the magnetic field generated by the current flowing around loop 2.



The appropriate force equation is

$$\bar{F}_{1,Bott} = \frac{\mu_0 I_1 I_2}{4\pi} \int_{\substack{C_1 \\ (Bottom)}} \oint_{C_2} \frac{d\bar{l}_1 \times [d\bar{l}_2 \times (\bar{r}_1 - \bar{r}_2)]}{|\bar{r}_1 - \bar{r}_2|^3}$$

$$\bar{F}_{1,B} = \bar{F}_{\substack{1,Bott \\ 2,Bott}} + \bar{F}_{\substack{1,Bott \\ 2,Top}} + \bar{F}_{\substack{1,Bott \\ 2,Left}} + \bar{F}_{\substack{1,Bott \\ 2,Right}}$$

contour C_2 for loop 2 has 4 sides

<u>Find/define pieces of integrals</u>

$I_1 = 10A$, $I_2 = 15A$, $d\bar{l}_{1,B} = dx_1\,\hat{a}_x$ $(0 < x_1 < 1m)$, $\bar{r}_1 = x_1\,\hat{a}_x$

$d\bar{l}_{2,B} = dx_{2B}\,\hat{a}_x$ $(2 < x_{2B} < 3m)$, $\bar{r}_{2,B} = x_{2B}\,\hat{a}_x$

$d\bar{l}_{2,T} = dx_{2T}\,\hat{a}_x$ $(2 < x_{2T} < 3m)$, $\bar{r}_{2,T} = x_{2T}\,\hat{a}_x + 1\,\hat{a}_y$ (m)

$d\bar{l}_{2,L} = dy_{2L}\,\hat{a}_y$ $(0 < y_{2L} < 1m)$, $\bar{r}_{2,L} = 2\hat{a}_x + y_{2L}\,\hat{a}_y$ (m)

$d\bar{l}_{2,R} = dy_{2R}\,\hat{a}_y$ $(0 < y_{2R} < 1m)$, $\bar{r}_{2,R} = 3\hat{a}_x + y_{2R}\,\hat{a}_y$ (m)

ex. cont.

$$\bar{F}_{1,B} = \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^{1} \int_{x_{2B}=2}^{3} \frac{dx_1 \hat{a}_x \times \left[dx_{2B} \hat{a}_x \times (x_1 \hat{a}_x - x_{2B} \hat{a}_x) \right]}{\left| x_1 \hat{a}_x - x_{2B} \hat{a}_x \right|^3} \to 0 \quad (\hat{a}_x \times \hat{a}_x = 0)$$

$$+ \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^{1} \int_{x_{2,T}=3}^{2} \frac{dx_1 \hat{a}_x \times \left[dx_{2T} \hat{a}_x \times (x_1 \hat{a}_x - x_{2T} \hat{a}_x - \hat{a}_y) \right]}{\left| x_1 \hat{a}_x - x_{2T} \hat{a}_x - \hat{a}_y \right|^3}$$

$$+ \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^{1} \int_{y_{2L}=1}^{0} \frac{dx_1 \hat{a}_x \times \left[dy_{2L} \hat{a}_y \times (x_1 \hat{a}_x - 2\hat{a}_x - y_{2L} \hat{a}_y) \right]}{\left| x_1 \hat{a}_x - 2\hat{a}_x - y_{2L} \hat{a}_y \right|^3}$$

$$+ \frac{\mu_0 (10)(15)}{4\pi} \int_{x_1=0}^{1} \int_{y_{2R}=0}^{1} \frac{dx_1 \hat{a}_x \times \left[dy_{2R} \hat{a}_y \times (x_1 \hat{a}_x - 3\hat{a}_x - y_{2R} \hat{a}_y) \right]}{\left| x_1 \hat{a}_x - 3\hat{a}_x - y_{2R} \hat{a}_y \right|^3}$$

$$\bar{F}_{1B} = 0 + 1.5 \times 10^{-5} \int_{x_1=0}^{1} \int_{x_{2T}=3}^{2} \frac{dx_1 \hat{a}_x \times \left[0 - 0 - dx_{2T} \hat{a}_z \right]}{\left[(x_1 - x_{2T})^2 + (-1)^2 \right]^{3/2}}$$

$$+ 1.5 \times 10^{-5} \int_{x_1=0}^{1} \int_{y_{2L}=1}^{0} \frac{dx_1 \hat{a}_x \times \left[-x_1 dy_{2L} \hat{a}_z + 2 dy_{2L} \hat{a}_z + 0 \right]}{\left[(x_1 - 2)^2 + (-y_{2L})^2 \right]^{3/2}}$$

$$+ 1.5 \times 10^{-5} \int_{x_1=0}^{1} \int_{y_{2R}=0}^{1} \frac{dx_1 \hat{a}_x \times \left[-x_1 dy_{2R} \hat{a}_z + 3 dy_{2R} \hat{a}_z - 0 \right]}{\left[(x_1 - 3)^2 + (-y_{2R})^2 \right]^{3/2}}$$

ex. cont.

$$\bar{F}_{1B} = 1.5\times10^{-5} \int_{x_1=0}^{1} \int_{x_{2T}=3}^{2} \frac{+\hat{a}_y \, dx_{2T} \, dx_1}{\left[(x_1 - x_{2T})^2 + 1^2\right]^{3/2}}$$

$$+1.5\times10^{-5} \int_{x_1=0}^{1} \int_{y_{2L}=1}^{0} \frac{+\hat{a}_y (x_1 - 2) \, dy_{2L} \, dx_1}{\left[(x_1 - 2)^2 + y_{2L}^2\right]^{3/2}}$$

$$+1.5\times10^{-5} \int_{x_1=0}^{1} \int_{y_{2R}=0}^{1} \frac{+\hat{a}_y (x_1 - 3) \, dy_{2R} \, dx_1}{\left[(x_1 - 3)^2 + y_{2R}^2\right]^{3/2}}$$

Method 1 – use MathCad to Solve integrals

$$\int_0^1 \int_3^2 \frac{1}{\left[(x1 - x2T)^2 + 1^2\right]^{1.5}} \, dx2T \, dx1 = -0.1043553$$

$$\int_0^1 \int_1^0 \frac{(x1 - 2)}{\left[(x1 - 2)^2 + y2L^2\right]^{1.5}} \, dy2L \, dx1 = 0.4001618$$

$$\int_0^1 \int_0^1 \frac{(x1 - 3)}{\left[(x1 - 3)^2 + y2R^2\right]^{1.5}} \, dy2R \, dx1 = -0.1537617$$

ex. cont.

Method 1 cont.

$$\bar{F}_{1B} = \hat{a}_y\, 1.5 \times 10^{-5} \left[-0.1043553 + 0.4001618 - 0.1537617 \right]$$

$$\boxed{\bar{F}_{1B} = \hat{a}_y\, 2.1307\ \mu N}$$

Method 2 – solve integrals analytically

use $\displaystyle\int \frac{dx}{\left[(a-x)^2+b^2\right]^{3/2}} = \frac{x-a}{b^2\sqrt{(a-x)^2+b^2}}$ ← 1st term wrt x_{2T}

and $\displaystyle\int \frac{(x-a)\,dx}{\left[(x-a)^2+b^2\right]^{3/2}} = \frac{-1}{\sqrt{(x-a)^2+b^2}}$ ← 2nd & 3rd terms wrt x_1

$$\bar{F}_{1B} = \hat{a}_y\, 1.5 \times 10^{-5} \int_{x_1=0}^{1} \left(\frac{x_{2T}-x_1}{1^2\sqrt{(x_1-x_{2T})^2+1^2}} \right)\Bigg|_{x_{2T}=3}^{2} dx_1$$

$$+\, \hat{a}_y\, 1.5 \times 10^{-5} \int_{y_{2L}=1}^{0} \left(\frac{-1}{\sqrt{(x_1-2)^2+y_{2L}^2}} \right)\Bigg|_{x_1=0}^{1} dy_{2L}$$

$$+\, \hat{a}_y\, 1.5 \times 10^{-5} \int_{y_{2R}=0}^{1} \left(\frac{-1}{\sqrt{(x_1-3)^2+y_{2R}^2}} \right)\Bigg|_{x_1=0}^{1} dy_{2R}$$

ex. cont.

Method 2 cont.

$$\bar{F}_{1B} = \hat{a}_y\, 1.5\times10^{-5} \int_{x_1=0}^{1} \left(\frac{(2-x_1)}{\sqrt{(x_1-2)^2+1^2}} - \frac{3-x_1}{\sqrt{(x_1-3)^2+1^2}} \right) dx_1$$

$$+ \hat{a}_y\, 1.5\times10^{-5} \int_{y_{2L}=1}^{0} \left[\frac{-1}{\sqrt{y_{2L}^2+1^2}} - \frac{-1}{\sqrt{y_{2L}^2+2^2}} \right] dy_{2L}$$

$$+ \hat{a}_y\, 1.5\times10^{-5} \int_{y_{2R}=0}^{1} \left[\frac{-1}{\sqrt{y_{2R}^2+2^2}} - \frac{-1}{\sqrt{y_{2R}^2+3^2}} \right] dy_{2R}$$

Now, use $$\int \frac{(x-a)}{\sqrt{(x-a)^2+b^2}}\, dx = \sqrt{(x-a)^2+b^2}$$

and $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\bar{F}_{1B} = \hat{a}_y\, 1.5\times10^{-5} \left[-\sqrt{(x_1-2)^2+1^2} + \sqrt{(x_1-3)^2+1^2} \right] \Bigg|_{x_1=0}^{1}$$

$$+ \hat{a}_y\, 1.5\times10^{-5} \left[-\ln\left(y_{2L}+\sqrt{y_{2L}^2+1^2}\right) + \ln\left(y_{2L}+\sqrt{y_{2L}^2+2^2}\right) \right] \Bigg|_{y_{2L}=1}^{0}$$

$$+ \hat{a}_y\, 1.5\times10^{-5} \left[-\ln\left(y_{2R}+\sqrt{y_{2R}^2+2^2}\right) + \ln\left(y_{2R}+\sqrt{y_{2R}^2+3^2}\right) \right] \Bigg|_{y_{2R}=0}^{1}$$

ex. cont.

Method 2 cont.

$$\bar{F}_{1B} = \hat{a}_y\, 1.5\times10^{-5} \left[\left(-\underbrace{\sqrt{(1-2)^2+1^2}}_{-\sqrt{2}} + \underbrace{\sqrt{(0-2)^2+1^2}}_{\sqrt{5}} \right) + \left(\underbrace{\sqrt{(1-3)^2+1^2}}_{\sqrt{5}} - \underbrace{\sqrt{(0-3)^2+1^2}}_{-\sqrt{10}} \right) \right]$$

$$+ \hat{a}_y\, 1.5\times10^{-5} \left[\left(-\ln\big(\underbrace{0+\sqrt{0+1}}_{\to 0}\big) + \ln\left(1+\sqrt{1^2+1^2}\right) \right) + \left(\ln\big(\underbrace{0+\sqrt{0+2^2}}_{\to \ln 2}\big) - \ln\left(1+\sqrt{1^2+2^2}\right) \right) \right]$$

$$+ \hat{a}_y\, 1.5\times10^{-5} \left[\left(-\ln\left(1+\sqrt{1^2+2^2}\right) + \ln\big(\underbrace{0+\sqrt{0+2^2}}_{\to \ln 2}\big) \right) + \left(\ln\left(1+\sqrt{1^2+3^2}\right) - \ln\big(\underbrace{0+\sqrt{0+3^2}}_{\to \ln 3}\big) \right) \right]$$

$$= \hat{a}_y\, 1.5\times10^{-5}(-0.1043553)$$

$$+ \hat{a}_y\, 1.5\times10^{-5}(0.40016176)$$

$$+ \hat{a}_y\, 1.5\times10^{-5}(-0.1537617)$$

Same integral results as calculated by MathCad, just lots more work

$$\boxed{\bar{F}_{1B} = \hat{a}_y\, 2.1307\ \mu N}$$ ← Same answer

⇒ To find complete force on loop 1 would require repeating this process for the other 3 sides!