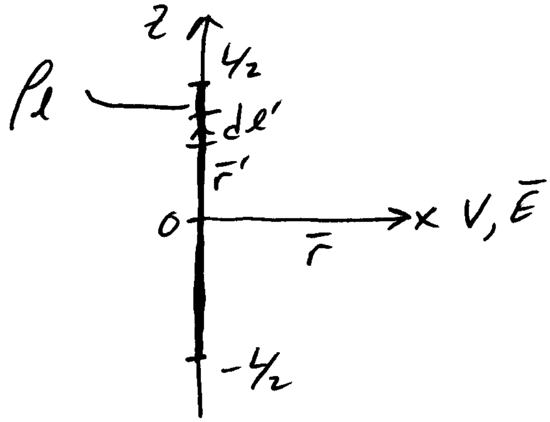
ex. Electric field due to finite length uniform line charge.



$$\bar{r}' = z'\hat{a}_z$$

$$\bar{r} = \rho\,\hat{a}_\rho$$

$$d\ell' = dz'$$

$$\rho_\ell(\bar{r}') = \rho_\ell$$

Earlier, we found that the voltage/potential was

$$V = \frac{\rho_\ell}{4\pi\epsilon_0}\ln\left[\frac{L/2 + \sqrt{(L/2)^2+\rho^2}}{-L/2 + \sqrt{(L/2)^2+\rho^2}}\right]$$

Using V

$$\bar{E} = -\bar{\nabla}V = -\hat{a}_\rho\frac{\partial V}{\partial\rho} - \hat{a}_\phi\frac{1}{\rho}\underbrace{\cancel{\frac{\partial V}{\partial\phi}}}_{0} - \hat{a}_z\underbrace{\cancel{\frac{\partial V}{\partial z}}}_{0}$$

$$= -\hat{a}_\rho\frac{\rho_\ell}{4\pi\epsilon_0}\frac{\partial}{\partial\rho}\left\{\ln\left[\frac{L/2+\sqrt{(L/2)^2+\rho^2}}{-L/2+\sqrt{(L/\rho)^2+\rho^2}}\right]\right\}$$

$$= -\hat{a}_\rho\frac{\rho_\ell}{4\pi\epsilon_0}\frac{\partial}{\partial\rho}\left\{\ln\left(L/2+\sqrt{(L/2)^2+\rho^2}\right) - \ln\left(-L/2+\sqrt{(L/2)^2+\rho^2}\right)\right\}$$

$$= -\hat{a}_\rho\frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{1}{L/2+\sqrt{(L/2)^2+\rho^2}}\,(1/2)\left[(L/2)^2+\rho^2\right]^{-1/2}(2\rho)\right.$$

$$\left. - \frac{1}{-L/2+\sqrt{(L/2)^2+\rho^2}}\,(1/2)\left[(L/2)^2+\rho^2\right]^{-1/2}(2\rho)\right]$$

ex. cont.

$$\bar{E} = -\hat{a}_\rho \frac{\rho_\ell}{4\pi\epsilon_0} \frac{\rho}{\sqrt{(L/2)^2+\rho^2}} \left[\frac{\left(-L/2+\sqrt{(L/2)^2+\rho^2}\right) - \left(L/2+\sqrt{(L/2)^2+\rho^2}\right)}{\left(L/2+\sqrt{(L/2)^2+\rho^2}\right)\left(-L/2+\sqrt{(L/2)^2+\rho^2}\right)} \right]$$

$$= -\hat{a}_\rho \frac{\rho_\ell}{4\pi\epsilon_0} \frac{\rho}{\sqrt{(L/2)^2+\rho^2}} \left[\frac{-L}{-(L/2)^2+0+(L/2)^2+\rho^2} \right]$$

$$\boxed{\bar{E} = +\hat{a}_\rho \frac{\rho_\ell}{4\pi\epsilon_0} \frac{L}{\rho\sqrt{(L/2)^2+\rho^2}} \quad \left(V/m\right)}$$

Calculating $\bar{E}$ directly gives:

$$\bar{E} = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_\ell(\bar{r}')(\bar{r}-\bar{r}')\,d\ell'}{|\bar{r}-\bar{r}'|^3}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{z'=-L/2}^{L/2} \frac{\rho_\ell(\rho\hat{a}_\rho - z'\hat{a}_z)}{|\rho\hat{a}_\rho - z'\hat{a}_z|^3}\,dz' \quad \leftarrow \text{No primed unit vectors!}$$

$$= \frac{\rho_\ell}{4\pi\epsilon_0}\left[\rho\hat{a}_\rho \int_{z'=-L/2}^{L/2} \frac{dz'}{[\rho^2+z'^2]^{3/2}} - \hat{a}_z \int_{z'=-L/2}^{L/2} \frac{z'\,dz'}{[\rho^2+z'^2]^{3/2}} \right]$$

$$= \frac{\rho_\ell}{4\pi\epsilon_0}\left[\rho\hat{a}_\rho \left.\frac{z'}{\rho^2\sqrt{\rho^2+z'^2}}\right|_{z'=-L/2}^{L/2} + \hat{a}_z \underbrace{\left.\left(\frac{1}{\sqrt{\rho^2+z'^2}}\right)\right|_{z'=-L/2}^{L/2}}_{\to 0} \right]$$

$$\boxed{\bar{E} = \hat{a}_\rho \frac{\rho_\ell}{4\pi\epsilon_0} \frac{L}{\rho\sqrt{\rho^2+(L/2)^2}}}$$

∴ ← Same answer!