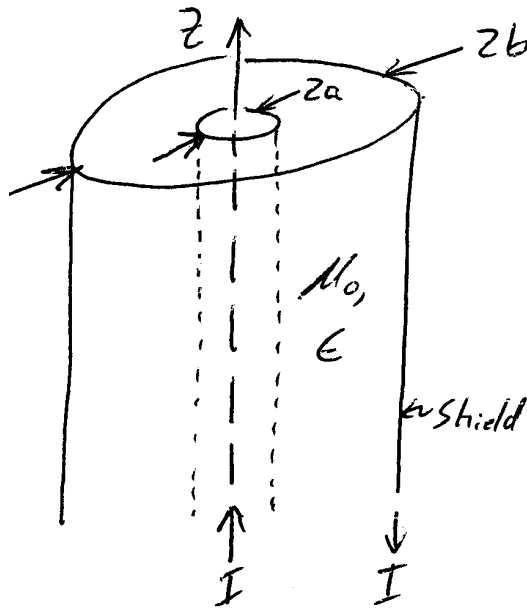


Example- Find the DC/low-frequency self-inductance per-unit-length of a coaxial transmission line.

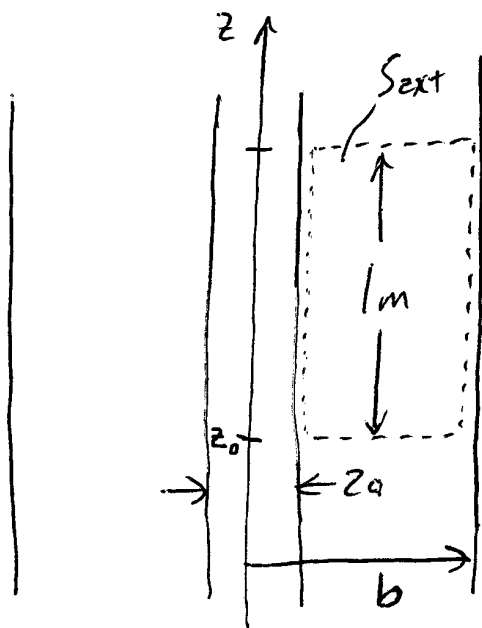


Using Ampere's Law, $\oint \vec{H} \cdot d\vec{l} = I_{enc}$,

$$\vec{B} = \begin{cases} \hat{a}_\phi \frac{\mu_0 I \rho}{2\pi a^2} & \rho \leq a \\ \hat{a}_\phi \frac{\mu_0 I}{2\pi \rho} & a < \rho < b \\ 0 & \rho > b \end{cases}$$

Start by finding the flux & flux linkage in the region $a < \rho < b$

Side View



$$\Psi_{ext} = \iint_{S_{ext}} \vec{B} \cdot d\vec{S}_\phi = \int_{z=z_0}^{z_0+l} \int_{\rho=a}^{\rho=b} \hat{a}_\phi \frac{\mu_0 I}{2\pi \rho} \cdot \hat{a}_\phi d\rho dz$$

$$= \frac{\mu_0 I}{2\pi} \int_{z_0}^{z_0+l} dz \int_{\rho=a}^{\rho=b} \frac{d\rho}{\rho}$$

$$= \frac{\mu_0 I}{2\pi} z \Big|_{z_0}^{z_0+l} \ln \rho \Big|_a^b$$

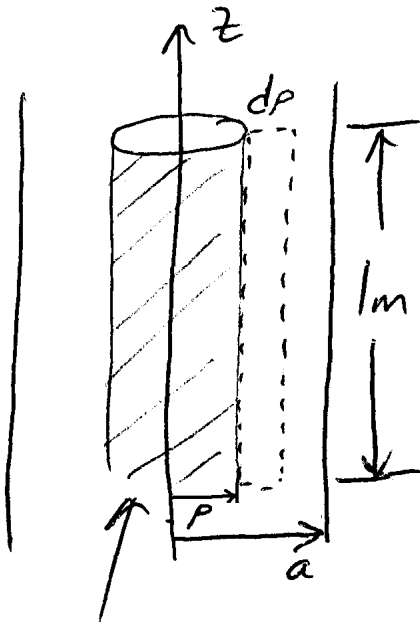
$$\Psi_{ext} = \frac{\mu_0 I}{2\pi} (l) \ln(b/a)$$

No multiple loops of wire and all of I is used to generate \vec{B} for $a < \rho < b$.

$$\frac{\Psi_{\text{ext}}}{(1\text{m})} = \frac{\lambda_{\text{ext}}}{(1\text{m})} = \frac{\mu_0 I}{2\pi} \ln(b/a)$$

Next, find the flux and flux linkage in the region $\rho < a$.

→ Here only a portion of the current is linked to the flux at any particular ρ



only current in this volume is responsible/generates magnetic flux thru $ds = dp(1\text{m})$

What fraction of current is within a radius ρ ?

$$\frac{I(\rho)}{I} = \frac{J_0(\pi\rho^2)}{J_0(\pi a^2)} = \frac{\rho^2}{a^2}$$

↳ assumes uniform current density (reasonable for DC current)

The differential amount of flux through dS is

$$d\psi_{\text{int}} = \vec{B} \cdot d\vec{S}_\phi = \hat{a}_\phi \frac{\mu_0 I \rho}{2\pi a^2} \cdot \hat{a}_\phi d\rho (lm) \quad \leftarrow dz$$

$$= \frac{\mu_0 I \rho}{2\pi a^2} (lm) d\rho$$

$$\frac{d\psi_{\text{int}}}{(lm)} = \frac{\mu_0 I \rho}{2\pi a^2} d\rho$$

Now, this flux is only linked to $\frac{\rho^2}{a^2}$ of the overall current I . So,

$$\frac{d\lambda_{\text{int}}}{(lm)} = \frac{\mu_0 I \rho}{2\pi a^2} d\rho \left(\frac{\rho^2}{a^2}\right) = \frac{\mu_0 I \rho^3}{2\pi a^4} d\rho$$

$$\frac{\lambda_{\text{int}}}{(lm)} = \int_{\rho=0}^a \frac{\mu_0 I \rho^3}{2\pi a^4} d\rho = \frac{\mu_0 I}{2\pi a^4} \left. \frac{\rho^4}{4} \right|_0^a = \frac{\mu_0 I}{8\pi}$$

$$\frac{\lambda_{\text{TOT}}}{(lm)} = \frac{\lambda_{\text{ext}}}{(lm)} + \frac{\lambda_{\text{int}}}{(lm)} = \frac{\mu_0 I}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_0 I}{8\pi}$$

$$\frac{L_{\text{TOT}}}{(lm)} = \frac{\lambda_{\text{TOT}} / (lm)}{I} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) + \frac{\mu_0}{8\pi} \quad (H/m)$$

How significant is the contribution from the internal flux linkage?

50 Ω air-filled coaxial line ($b/a = 2.3$)

$$\begin{aligned} \frac{L_{TOT}}{(1m)} &= \frac{4\pi \times 10^{-7}}{2\pi} \ln(2.3) + \frac{4\pi \times 10^{-7}}{8\pi} \\ &= (2 \times 10^{-7}) \ln(2.3) + 0.5 \times 10^{-7} \\ &\quad \uparrow \qquad \qquad \uparrow \\ &\quad L_{ext} \qquad \quad L_{int} \end{aligned}$$

$$= 2.16582 \times 10^{-7} \text{ (H/m)}$$

$$\frac{L_{int}}{L_{TOT}} = \frac{0.5 \times 10^{-7}}{2.1658 \times 10^{-7}} \times 100\% = \underline{\underline{23.1\%}}$$

R6-6 75 Ω coaxial line ($b/a = 4.25$)

$$\begin{aligned} \frac{L_{TOT}}{(1m)} &= \frac{4\pi \times 10^{-7}}{2\pi} \ln(4.25) + \frac{4\pi \times 10^{-7}}{8\pi} \\ &= 2 \times 10^{-7} \ln(4.25) + 0.5 \times 10^{-7} \\ &= 3.394 \times 10^{-7} \text{ (H/m)} \end{aligned}$$

$$\frac{L_{int}}{L_{TOT}} = \frac{0.5 \times 10^{-7}}{3.394 \times 10^{-7}} \times 100\% = \underline{\underline{14.7\%}}$$