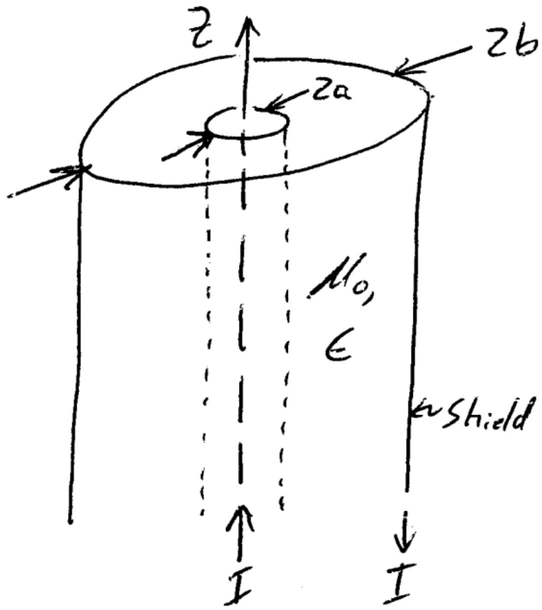
Example- Find the per-unit-length self-inductance of a coaxial transmission line.

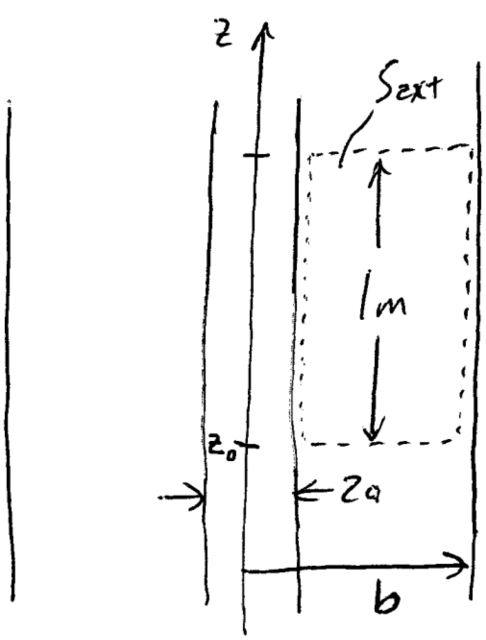


Using Ampere's Law, $\oint \bar{H}\cdot d\bar{\ell} = I_{enc}$,

$$\bar{B} = \begin{cases} \hat{a}_\phi \dfrac{\mu_0 I \rho}{2\pi a^2} & \rho \le a \\ \hat{a}_\phi \dfrac{\mu_0 I}{2\pi \rho} & a < \rho < b \\ 0 & \rho > b \end{cases}$$

Start by finding the flux & flux linkage in the region $a < \rho < b$

Side View



$$\psi_{ext} = \iint_{S_{ext}} \bar{B}\cdot d\bar{S}_\phi = \int_{z=z_0}^{z_0+l} \int_{\rho=a}^{b} \hat{a}_\phi \frac{\mu_0 I}{2\pi\rho} \cdot \hat{a}_\phi \, d\rho \, dz$$

$$= \frac{\mu_0 I}{2\pi} \int_{z_0}^{z_0+l} dz \int_{\rho=a}^{b} \frac{d\rho}{\rho}$$

$$= \frac{\mu_0 I}{2\pi} \, z \Big|_{z_0}^{z_0+l} \ln\rho \Big|_a^b$$

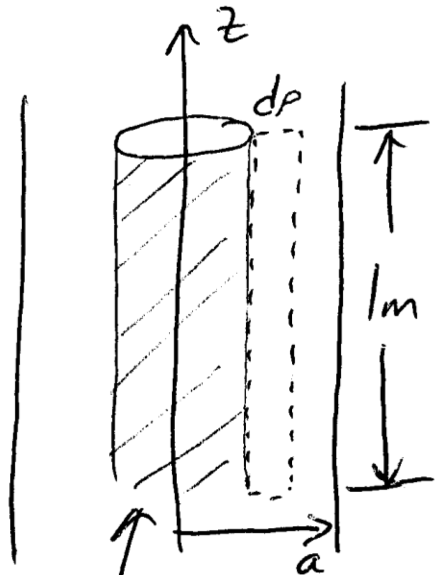
$$\psi_{ext} = \frac{\mu_0 I}{2\pi} (l) \ln(b/a)$$

No multiple loops of wire and all of I is used to generate $\bar{B}$ for $a<\rho<b$.

$$\frac{\psi_{ext}}{(1m)} = \frac{\lambda_{ext}}{(1m)} = \frac{\mu_0 I}{2\pi} \ln(b/a)$$

Next, find the flux and flux linkage in the region $\rho<a$.

→ Here only a portion of the current is linked to the flux at any particular ρ



only current in this volume is responsible/generates magnetic flux thru $ds = d\rho(1m)$

What fraction of current is within a radius ρ?

$$\frac{I(\rho)}{I} = \frac{J_0(\pi\rho^2)}{J_0(\pi a^2)} = \frac{\rho^2}{a^2}$$

assumes uniform current density (reasonable for DC current)

The differential amount of flux through dS is

$$d\Psi_{int} = \overline{B} \cdot d\overline{s}_\phi = \hat{a}_\phi \frac{\mu_0 I \rho}{2\pi a^2} \cdot \hat{a}_\phi \, d\rho \, (\overset{dz}{1m})$$

$$= \frac{\mu_0 I \rho}{2\pi a^2} (1m) d\rho$$

$$\frac{d\Psi_{int}}{(1m)} = \frac{\mu_0 I \rho}{2\pi a^2} d\rho$$

Now, this flux is only linked to ρ^2/a^2 of the overall current I. So,

$$\frac{d\lambda_{int}}{(1m)} = \frac{\mu_0 I \rho}{2\pi a^2} d\rho \left(\rho^2/a^2\right) = \frac{\mu_0 I \rho^3}{2\pi a^4} d\rho$$

$$\frac{\lambda_{int}}{(1m)} = \int_{\rho=0}^{a} \frac{\mu_0 I \rho^3}{2\pi a^4} d\rho = \frac{\mu_0 I}{2\pi a^4} \left. \frac{\rho^4}{4} \right|_0^a = \underline{\frac{\mu_0 I}{8\pi}}$$

$$\frac{\lambda_{TOT}}{(1m)} = \frac{\lambda_{ext}}{(1m)} + \frac{\lambda_{int}}{(1m)} = \frac{\mu_0 I}{2\pi} \ln\left(b/a\right) + \frac{\mu_0 I}{8\pi}$$

$$\underline{\underline{\frac{L_{TOT}}{(1m)} = \frac{\lambda_{TOT}/(1m)}{I} = \frac{\mu_0}{2\pi} \ln\left(b/a\right) + \frac{\mu_0}{8\pi} \quad \left(H/m\right)}}$$

How significant is the contribution from the internal flux linkage?

<u>$50\,\Omega$ air-filled coaxial line</u> $\left(b/a \approx 2.3\right)$

$$\frac{L_{TOT}}{(1m)} = \frac{4\pi \times 10^{-7}}{2\pi}\ln(2.3) + \frac{4\pi \times 10^{-7}}{8\pi}$$

$$= \underbrace{\left(2\times10^{-7}\right)\ln(2.3)}_{L_{ext}} + \underbrace{0.5\times10^{-7}}_{L_{int}}$$

$$= 2.16582\times10^{-7}\ (H/m)$$

$$\frac{L_{int}}{L_{TOT}} = \frac{0.5\times10^{-7}}{2.1658\times10^{-7}}\times100\% = \underline{\underline{23.1\%}}$$

<u>RG-6 $75\,\Omega$ coaxial line</u> $\left(b/a \approx 4.25\right)$

$$\frac{L_{TOT}}{(1m)} = \frac{4\pi\times10^{-7}}{2\pi}\ln(4.25) + \frac{4\pi\times10^{-7}}{8\pi}$$

$$= 2\times10^{-7}\ln(4.25) + 0.5\times10^{-7}$$

$$= 3.394\times10^{-7}\ (H/m)$$

$$\frac{L_{int}}{L_{TOT}} = \frac{0.5\times10^{-7}}{3.394\times10^{-7}}\times100\% = \underline{\underline{14.7\%}}$$