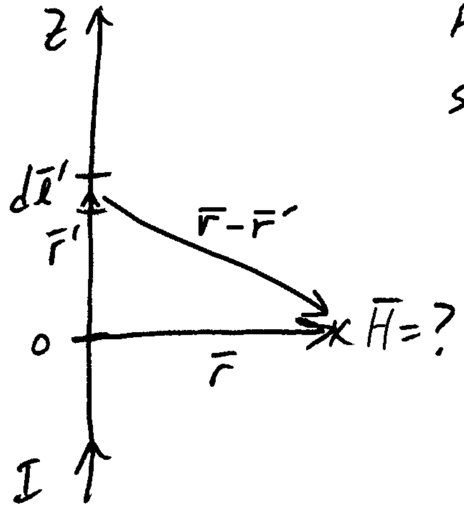
ex. Find magnetic field vector due to current I flowing along z-axis adjacent to origin.



First, look at a differential section using cylindrical coord.

$$d\bar{H} = \frac{I\,d\bar{l}' \times (\bar{r}-\bar{r}')}{4\pi|\bar{r}-\bar{r}'|^3}$$

$$d\bar{l}' = \hat{a}_{\rho'}\underbrace{d\rho'}_{\to 0} + \underbrace{\rho'}_{\to 0} d\phi'\,\hat{a}_{\phi'} + dz'\,\hat{a}_z = dz'\,\hat{a}_z$$

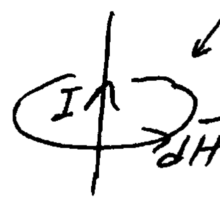
$$\bar{r} = \rho\,\hat{a}_\rho + \underbrace{z}_{\to 0}\hat{a}_z = \rho\,\hat{a}_\rho$$

$$\bar{r}' = \underbrace{\rho'}_{\to 0}\hat{a}_{\rho'} + z'\,\hat{a}_z = z'\,\hat{a}_z$$

$$d\bar{H} = \frac{I\,dz'\,\hat{a}_z \times (\rho\,\hat{a}_\rho - z'\,\hat{a}_z)}{4\pi|\rho\,\hat{a}_\rho - z'\,\hat{a}_z|^3}$$

$$= \frac{I\,dz'\,\rho\,\hat{a}_\phi - 0}{4\pi\left(\sqrt{\rho^2+(-z')^2}\right)^3}$$

$$\boxed{d\bar{H} = \hat{a}_\phi \frac{I}{4\pi}\frac{\rho\,dz'}{(\rho^2+z'^2)^{3/2}}}$$



$\bar{H}$ forms closed loops around current element (IEEE Symbol)

Next, find $\bar{H}$ due a length L (centered on the origin) of this current I

$$\bar{H} = \int_{C'} \frac{I\,d\bar{l}' \times (\bar{r}-\bar{r}')}{4\pi|\bar{r}-\bar{r}'|^3} = \hat{a}_\phi \frac{I\rho}{4\pi}\int_{z'=-L/2}^{L/2} \frac{dz'}{(\rho^2+z'^2)^{3/2}}$$

$$= \hat{a}_\phi \frac{I\rho}{4\pi} \left. \frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \right|_{z'=-L/2}^{L/2}$$

$$= \hat{a}_\phi \frac{I}{4\pi\rho} \left[\frac{L/2}{\sqrt{\rho^2 + (L/2)^2}} - \frac{-L/2}{\sqrt{\rho^2 + (-L/2)^2}} \right]$$

$$\boxed{\bar{H} = \hat{a}_\phi \frac{I}{2\pi\rho} \frac{L/2}{\sqrt{\rho^2 + L^2/4}}} \quad @ \; z = 0$$

Last, what if we assume the wire is infinitely long?

$$\bar{H} = \oint_{c'} \frac{I\, d\bar{\ell}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3} = \hat{a}_\phi \frac{I\rho}{4\pi} \int_{z'=-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}}$$

$$= \hat{a}_\phi \frac{I}{4\pi\rho} \left. \frac{z'}{\sqrt{\rho^2 + z'^2}} \right|_{z'=-\infty}^{\infty}$$

$$= \hat{a}_\phi \frac{I}{4\pi\rho} \left[\underbrace{\frac{\infty}{\infty} - \frac{-\infty}{\infty}}_{\to 2} \right]$$

$$\boxed{\bar{H} = \hat{a}_\phi \frac{I}{2\pi\rho}}$$

very important for visualizing problems