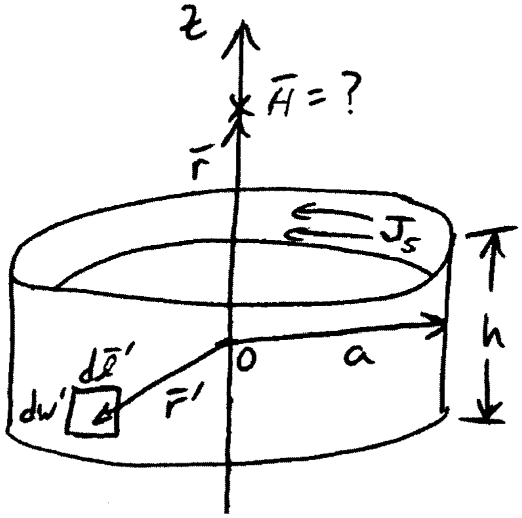
ex. A circular ring of radius a and height h is centered on the z-axis at the origin as shown. If it carries a uniform current density J_s, find the magnetic field vector along the z-axis.



$\Rightarrow$ use cylindrical coordinates

$\Rightarrow$ use surface current density form of Biot-Savart's Law

$$\bar{H} = \oint_{c'} \int_{w'} \frac{J_s \, dw' \, d\bar{l}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$$

$$dw' = dz', \quad d\bar{l}' = \underbrace{d\rho'}_{\to 0} \hat{a}_{\rho'} + \underbrace{\rho'}_{\to a} d\phi' \hat{a}_{\phi'} + \underbrace{dz'}_{\to 0} \hat{a}_z = a \, d\phi' \hat{a}_{\phi'}$$

constant radius a

$$\bar{r} = \underbrace{\rho}_{\to 0} \hat{a}_\rho + z \hat{a}_z = z \hat{a}_z \quad \text{(on z-axis)}$$

$$\bar{r}' = \underbrace{\rho'}_{\to a} \hat{a}_{\rho'} + z' \hat{a}_z = a \hat{a}_{\rho'} + z' \hat{a}_z$$

$$\bar{H} = \frac{J_s}{4\pi} \int_{\phi'=0}^{2\pi} \int_{z'=-h/2}^{h/2} \frac{dz' \, a \, d\phi' \, \hat{a}_{\phi'} \times (z \hat{a}_z - a \hat{a}_{\rho'} - z' \hat{a}_z)}{|z \hat{a}_z - a \hat{a}_{\rho'} - z' \hat{a}_z|^3}$$

ex. cont.

In order to compute the cross products and integrals, convert all the unit vectors to rectangular coordinates

$$\hat{a}_{\phi'} = -\sin\phi'\,\hat{a}_x + \cos\phi'\,\hat{a}_y \qquad \hat{a}_{\rho'} = \cos\phi'\,\hat{a}_x + \sin\phi'\,\hat{a}_y$$

$$\bar{H} = \frac{a J_s}{4\pi}\int_{\phi'=0}^{2\pi}\int_{z'=-\frac{h}{2}}^{h/2} \frac{dz'\,d\phi'\,(-\sin\phi'\hat{a}_x + \cos\phi'\hat{a}_y)\times(-a\cos\phi'\hat{a}_x - a\sin\phi'\hat{a}_y + (z-z')\hat{a}_z)}{|-a\cos\phi'\hat{a}_x - a\sin\phi'\hat{a}_y + (z-z')\hat{a}_z|^3}$$

$$= \frac{a J_s}{4\pi}\int_{\phi'=0}^{2\pi}\int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \frac{dz'\,d\phi'\left[0 + a\sin^2\phi'\hat{a}_z + (z-z')\sin\phi'\hat{a}_y + a\cos^2\phi'\hat{a}_z + 0 + (z-z')\cos\phi'\hat{a}_x\right]}{\left[(-a\cos\phi')^2 + (-a\sin\phi')^2 + (z-z')^2\right]^{3/2}}$$

→ Use trigonometric identity $\cos^2 A + \sin^2 A = 1$, break the integral into vector components, + evaluate $d\phi'$ first.

$$\bar{H} = \frac{a J_s}{4\pi}\int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \left\{ \hat{a}_x \frac{z-z'}{[(z-z')^2+a^2]^{3/2}} \underbrace{\int_{\phi'=0}^{2\pi}\cos\phi'\,d\phi'}_{\to\, 0 \text{ (full cycle)}} + \hat{a}_y \frac{z-z'}{[(z-z')^2+a^2]^{3/2}} \underbrace{\int_{\phi'=0}^{2\pi}\sin\phi'\,d\phi'}_{\to\, 0 \text{ (full cycle)}} + \hat{a}_z \frac{a}{[(z-z')^2+a^2]^{3/2}} \int_{\phi'=0}^{2\pi}(\underbrace{\cos^2\phi' + \sin^2\phi'}_{1})\,d\phi' \right\} dz'$$

ex. cont.

$$\bar{H} = \frac{a J_s}{4\pi} \int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \left\{ 0 + 0 + \hat{a}_z \frac{a}{[(z-z')^2+a^2]^{3/2}} (\phi')\Big|_0^{2\pi} \right\} dz'$$

$$= \hat{a}_z \frac{a^2 J_s (2\pi - 0)}{4\pi} \int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \frac{dz'}{[(z-z')^2 + a^2]^{3/2}}$$

look-up or do a change of variables on $\int \frac{du}{[u^2+a^2]^{3/2}}$

$$= \hat{a}_z \frac{a^2 J_s}{2} \left(\frac{-(z-z')}{a^2 \sqrt{(z-z')^2 + a^2}} \right) \Bigg|_{z'=-\frac{h}{2}}^{h/2}$$

$$\bar{H} = \hat{a}_z \frac{J_s}{2} \left[\frac{\frac{h}{2} - z}{\sqrt{(z-\frac{h}{2})^2 + a^2}} + \frac{z + h/2}{\sqrt{(z+\frac{h}{2})^2 + a^2}} \right]$$