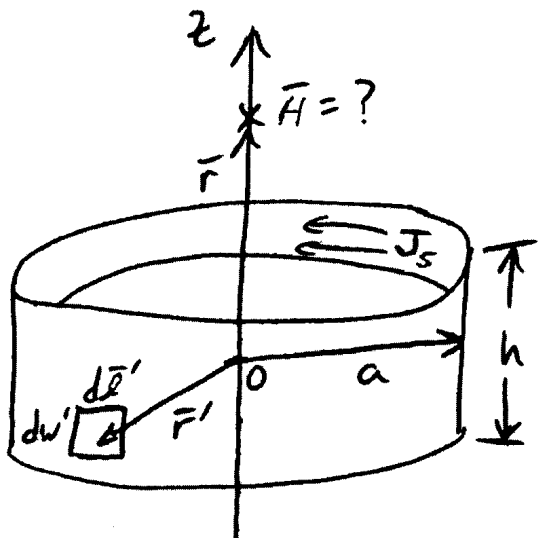


ex. A circular ring of radius a and height h is centered on the z -axis at the origin as shown. If it carries a uniform current density J_s , find the magnetic field vector along the z -axis.



⇒ use cylindrical coordinates

⇒ use surface current density form of Biot-Savart's Law

$$\bar{H} = \oint_{C'} \int_{w'} \frac{J_s dw' d\bar{l}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$$

$$dw' = dz', \quad d\bar{l}' = \underbrace{d\rho' \hat{a}_\rho'}_{\rightarrow 0} + \underbrace{\rho' d\phi' \hat{a}_\phi'}_{\rightarrow a} + \underbrace{dz' \hat{a}_z}_{\rightarrow 0} = a d\phi' \hat{a}_\phi'$$

constant radius a

$$\bar{r} = \underbrace{\rho \hat{a}_\rho}_{\rightarrow 0} + z \hat{a}_z = z \hat{a}_z \quad (\text{on } z\text{-axis})$$

$$\bar{r}' = \underbrace{\rho' \hat{a}_\rho'}_{\rightarrow a} + z' \hat{a}_z = a \hat{a}_\rho' + z' \hat{a}_z$$

$$\bar{H} = \frac{J_s}{4\pi} \int_{\phi'=0}^{2\pi} \int_{z'=-h/2}^{h/2} \frac{dz' a d\phi' \hat{a}_\phi' \times (z \hat{a}_z - a \hat{a}_\rho' - z' \hat{a}_z)}{|z \hat{a}_z - a \hat{a}_\rho' - z' \hat{a}_z|^3}$$

ex. cont.

In order to compute the cross products and integrals, convert all the unit vectors to rectangular coordinates

$$\hat{a}_{\phi'} = -\sin\phi' \hat{a}_x + \cos\phi' \hat{a}_y \quad \hat{a}_{\rho'} = \cos\phi' \hat{a}_x + \sin\phi' \hat{a}_y$$

$$\vec{H} = \frac{a J_s}{4\pi} \int_{\phi'=0}^{2\pi} \int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \frac{dz' d\phi' (-\sin\phi' \hat{a}_x + \cos\phi' \hat{a}_y) \times (-a \cos\phi' \hat{a}_x - a \sin\phi' \hat{a}_y + (z-z') \hat{a}_z)}{|-a \cos\phi' \hat{a}_x - a \sin\phi' \hat{a}_y + (z-z') \hat{a}_z|^3}$$

$$= \frac{a J_s}{4\pi} \int_{\phi'=0}^{2\pi} \int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \frac{dz' d\phi' [0 + a \sin^2\phi' \hat{a}_z + (z-z') \sin\phi' \hat{a}_y + a \cos^2\phi' \hat{a}_z + 0 + (z-z') \cos\phi' \hat{a}_x]}{[(-a \cos\phi')^2 + (-a \sin\phi')^2 + (z-z')^2]^{3/2}}$$

→ Use trigonometric identity $\cos^2 A + \sin^2 A = 1$, break the integral into vector components, + evaluate $d\phi'$ first.

$$\vec{H} = \frac{a J_s}{4\pi} \int_{z'=-\frac{h}{2}}^{\frac{h}{2}} \left\{ \hat{a}_x \frac{z-z'}{[(z-z')^2 + a^2]^{3/2}} \int_{\phi'=0}^{2\pi} \cos\phi' d\phi' \right. \\ \left. + \hat{a}_y \frac{z-z'}{[(z-z')^2 + a^2]^{3/2}} \int_{\phi'=0}^{2\pi} \sin\phi' d\phi' \right. \\ \left. + \hat{a}_z \frac{a}{[(z-z')^2 + a^2]^{3/2}} \int_{\phi'=0}^{2\pi} (\cos^2\phi' + \sin^2\phi') d\phi' \right\} dz'$$

→ 0 (full cycle)

→ 0 (full cycle)

ex. cont.

$$\bar{H} = \frac{a J_s}{4\pi} \int_{z' = -\frac{h}{2}}^{\frac{h}{2}} \left\{ 0 + 0 + \hat{a}_z \frac{a}{[(z-z')^2 + a^2]^{3/2}} (\phi') \Big|_0^{2\pi} \right\} dz'$$

$$= \hat{a}_z \frac{a^2 J_s (2\pi - 0)}{4\pi} \int_{z' = -\frac{h}{2}}^{\frac{h}{2}} \frac{dz'}{[(z-z')^2 + a^2]^{3/2}}$$

look-up or
do a change
of variables
on $\int \frac{du}{(u^2 + a^2)^{3/2}}$

$$= \hat{a}_z \frac{a^2 J_s}{2} \left(\frac{-(z-z')}{a^2 \sqrt{(z-z')^2 + a^2}} \right) \Big|_{z' = -\frac{h}{2}}^{\frac{h}{2}}$$

$$\bar{H} = \hat{a}_z \frac{J_s}{2} \left[\frac{\frac{h}{2} - z}{\sqrt{(z - \frac{h}{2})^2 + a^2}} + \frac{z + \frac{h}{2}}{\sqrt{(z + \frac{h}{2})^2 + a^2}} \right]$$
