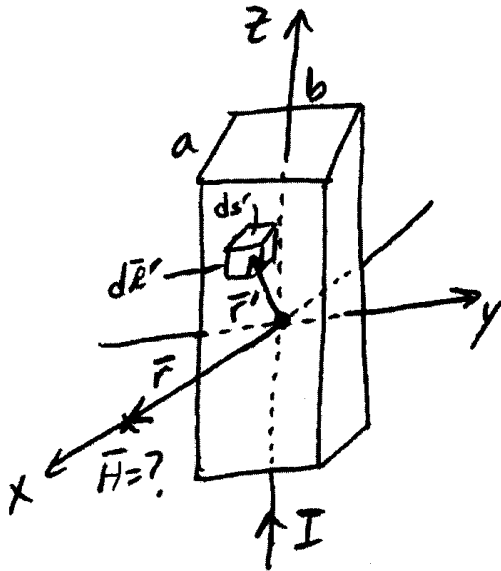


Example- Find the magnetic field along the x -axis for a very long rectangular bar ($a \times b$ cross section) centered on the z -axis in free space as shown. The bar carries a uniformly-distributed current I in the $+z$ -direction.



$$\vec{H} = \oint_{c'} \iint_{s'} \frac{J ds' d\vec{\ell}' \times (\vec{r} - \vec{r}')}{4\pi |\vec{r} - \vec{r}'|^3}$$

\Rightarrow use rectangular coordinates

$$J = \frac{\text{current}}{\text{area}} = \frac{I}{ab}$$

$$ds' = |d\vec{z}'| = dx' dy'$$

$$d\vec{\ell}' = dx' \hat{a}_x + dy' \hat{a}_y + dz' \hat{a}_z = dz' \hat{a}_z$$

\hookrightarrow in direction of current flow or c'

$$\vec{r} = x \hat{a}_x, \quad \vec{r}' = x' \hat{a}_x + y' \hat{a}_y + z' \hat{a}_z \quad \begin{matrix} \nearrow x \\ z \\ \nwarrow y \end{matrix}$$

$$\vec{H} = \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{y'=-b/2}^{b/2} \int_{x'=-a/2}^{a/2} \frac{dx' dy' dz' \hat{a}_z \times (x \hat{a}_x - x' \hat{a}_x - y' \hat{a}_y - z' \hat{a}_z)}{|x \hat{a}_x - x' \hat{a}_x - y' \hat{a}_y - z' \hat{a}_z|^3}$$

$$\vec{H} = \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{y'=-b/2}^{b/2} \int_{x'=-a/2}^{a/2} \frac{(x-x') \hat{a}_y + y' \hat{a}_x + 0}{[(x-x')^2 + (-y')^2 + (-z')^2]^{3/2}} dx' dy' dz'$$

\Rightarrow split into vector components + re-arrange order

ex. cont.

$$\vec{H} = \hat{a}_x \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} \int_{y'=-\frac{b}{2}}^{\frac{b}{2}} \frac{y' dy'}{[y'^2 + (x-x')^2 + z'^2]^{3/2}} dx' dz'$$

$$+ \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-\frac{b}{2}}^{\frac{b}{2}} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} \int_{z'=-\infty}^{\infty} \frac{dz'}{[(x-x')^2 + y'^2 + z'^2]^{3/2}} dx' dy'$$

$$= \hat{a}_x \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} \left[\frac{-1}{\sqrt{y'^2 + (x-x')^2 + z'^2}} \right]_{y'=-\frac{b}{2}}^{\frac{b}{2}} dx' dz'$$

$$+ \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-\frac{b}{2}}^{\frac{b}{2}} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} (x-x') \left[\frac{\frac{z'}{(x-x')^2 + y'^2}}{\sqrt{(x-x')^2 + y'^2 + z'^2}} \right]_{z'=-\infty}^{\infty} dx' dy'$$

$$= \hat{a}_x \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} \left[\frac{-1}{\sqrt{\frac{b^2}{4} + (x-x')^2 + z'^2}} + \frac{+1}{\sqrt{\frac{b^2}{4} + (x-x')^2 + z'^2}} \right] dx' dz'$$

→ 0 (no \hat{a}_x component)

$$+ \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-\frac{b}{2}}^{\frac{b}{2}} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} \frac{x-x'}{(x-x')^2 + y'^2} \left[\frac{\infty}{\infty} - \frac{-\infty}{\infty} \right] dx' dy'$$

→ 2

ex. cont.

$$\vec{H} = \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \int_{x'=-a/2}^{a/2} \frac{z(x-x') dx'}{(x-x')^2 + y'^2} dy'$$

adapt $\int \frac{u du}{u^2+a^2}$

$$= \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \left[-\ln((x-x')^2 + y'^2) \right] \Big|_{x'=-a/2}^{a/2} dy'$$

$$= \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \left[-\ln((x-a/2)^2 + y'^2) + \ln((x+a/2)^2 + y'^2) \right] dy'$$

Next, use $\int \ln(u^2+a^2) du = u \ln(u^2+a^2) - 2u + 2a \tan^{-1} \frac{u}{a}$

$$\vec{H} = \hat{a}_y \frac{I}{4\pi ab} \left[-y' \ln((x-a/2)^2 + y'^2) + \cancel{2y'} - 2(x-a/2) \tan^{-1} \left(\frac{y'}{x-a/2} \right) \right. \\ \left. + y' \ln((x+a/2)^2 + y'^2) - \cancel{2y'} + 2(x+a/2) \tan^{-1} \left(\frac{y'}{x+a/2} \right) \right] \Big|_{y'=-b/2}^{b/2}$$

cancel

$$= \hat{a}_y \frac{I}{4\pi ab} \left[-\frac{b}{2} \ln((x-a/2)^2 + \frac{b^2}{4}) + \frac{-b}{2} \ln((x-a/2)^2 + \frac{b^2}{4}) \right. \\ - 2(x-a/2) \tan^{-1} \left(\frac{b/2}{x-a/2} \right) + 2(x-a/2) \tan^{-1} \left(\frac{-b/2}{x-a/2} \right) \\ + \frac{b}{2} \ln((x+a/2)^2 + \frac{b^2}{4}) + \frac{b}{2} \ln((x+a/2)^2 + \frac{b^2}{4}) \\ \left. + 2(x+a/2) \tan^{-1} \left(\frac{b/2}{x+a/2} \right) - 2(x+a/2) \tan^{-1} \left(\frac{-b/2}{x+a/2} \right) \right]$$

Use Trigonometric identity $\tan^{-1}(-A) = -\tan^{-1}(A)$

ex. cont.

$$\vec{H} = \hat{a}_y \frac{I}{4\pi ab} \left[-b \ln\left((x - a/2)^2 + \frac{b^2}{4}\right) + b \ln\left((x + a/2)^2 + \frac{b^2}{4}\right) \right. \\ \left. - 4(x - a/2) \tan^{-1}\left(\frac{b/2}{x - a/2}\right) \right. \\ \left. + 4(x + a/2) \tan^{-1}\left(\frac{b/2}{x + a/2}\right) \right]$$

$$\vec{H} = \hat{a}_y \frac{I}{4\pi ab} \left[b \ln\left(\frac{(x + a/2)^2 + \frac{b^2}{4}}{(x - a/2)^2 + \frac{b^2}{4}}\right) - 4(x - a/2) \tan^{-1}\left(\frac{b/2}{x - a/2}\right) \right. \\ \left. + 4(x + a/2) \tan^{-1}\left(\frac{b/2}{x + a/2}\right) \right]$$

valid for $y = z = 0$

Test Biot-Savart solution to long rectangular bar problem

Select arbitrary values for bar dimensions & current and location on x -axis.

$$\mathbf{a} := 0.03 \text{ m} \quad \mathbf{b} := 0.02 \text{ m} \quad \mathbf{I} := 10 \text{ A} \quad \mathbf{x} := 0.04 \text{ m}$$

Numerically solve integrals for H_x and H_y components of the magnetic field vector.

$$\mathbf{Hx_num} := \frac{\mathbf{I}}{4 \cdot \pi \cdot \mathbf{a} \cdot \mathbf{b}} \cdot \int_{-\infty}^{\infty} \int_{-\frac{\mathbf{a}}{2}}^{\frac{\mathbf{a}}{2}} \int_{-\frac{\mathbf{b}}{2}}^{\frac{\mathbf{b}}{2}} \frac{\mathbf{yp}}{\left[(\mathbf{x} - \mathbf{xp})^2 + \mathbf{yp}^2 + \mathbf{zp}^2 \right]^{1.5}} \mathbf{dyp} \mathbf{d xp} \mathbf{d zp}$$

$$\mathbf{Hy_num} := \frac{\mathbf{I}}{4 \cdot \pi \cdot \mathbf{a} \cdot \mathbf{b}} \cdot \int_{-\infty}^{\infty} \int_{-\frac{\mathbf{b}}{2}}^{\frac{\mathbf{b}}{2}} \int_{-\frac{\mathbf{a}}{2}}^{\frac{\mathbf{a}}{2}} \frac{\mathbf{x} - \mathbf{xp}}{\left[(\mathbf{x} - \mathbf{xp})^2 + \mathbf{yp}^2 + \mathbf{zp}^2 \right]^{1.5}} \mathbf{d xp} \mathbf{d yp} \mathbf{d zp}$$

Calculate analytic solutions H_x and H_y components of the magnetic field vector .

$$\mathbf{Hxa} := 0$$

$$\mathbf{Hya} := \frac{\mathbf{I}}{4 \cdot \pi \cdot \mathbf{a} \cdot \mathbf{b}} \cdot \left[\mathbf{b} \cdot \ln \left[\frac{\left(\mathbf{x} + \frac{\mathbf{a}}{2} \right)^2 + \frac{\mathbf{b}^2}{4}}{\left(\mathbf{x} - \frac{\mathbf{a}}{2} \right)^2 + \frac{\mathbf{b}^2}{4}} \right] - 4 \cdot \left(\mathbf{x} - \frac{\mathbf{a}}{2} \right) \cdot \text{atan} \left(\frac{\frac{\mathbf{b}}{2}}{\mathbf{x} - \frac{\mathbf{a}}{2}} \right) \dots \right. \\ \left. + 4 \cdot \left(\mathbf{x} + \frac{\mathbf{a}}{2} \right) \cdot \text{atan} \left(\frac{\frac{\mathbf{b}}{2}}{\mathbf{x} + \frac{\mathbf{a}}{2}} \right) \right]$$

Compare analytic and numeric solutions.

$$\mathbf{Hx_num} = 0$$

$$\mathbf{Hy_num} = 40.76686 \text{ A/m}$$

$$\mathbf{Hxa} = 0$$

$$\mathbf{Hya} = 40.76686 \text{ A/m}$$

Perfect Match!