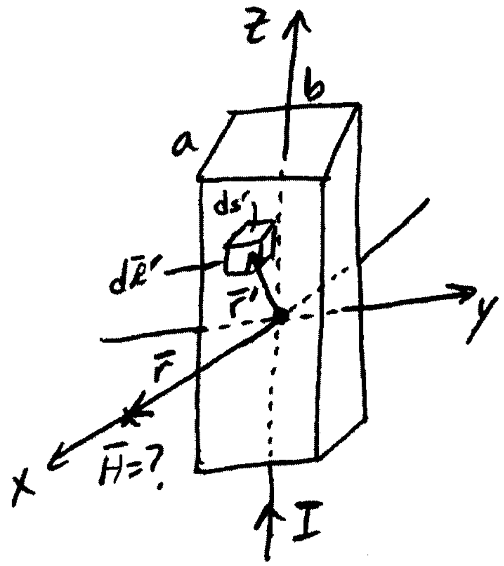
Example- Find the magnetic field along the x-axis due to a very long rectangular bar ($a \times b$ cross section) centered on the z-axis in free space as shown. The bar carries a uniformly-distributed current I in the $+z$-direction.



$$\bar{H} = \oint_{c'} \iint_{s'} \frac{J\, ds'\, d\bar{\ell}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3}$$

$\Rightarrow$ use rectangular coordinates

$$J = \frac{\text{current}}{\text{area}} = \frac{I}{ab}$$

$$ds' = |d\bar{s}_z'| = dx'dy'$$

$$d\bar{\ell}' = \cancelto{0}{dx'}\,\hat{a}_x + \cancelto{0}{dy'}\,\hat{a}_y + dz'\,\hat{a}_z = dz'\,\hat{a}_z$$

$$\bar{r} = x\,\hat{a}_x, \quad \bar{r}' = x'\hat{a}_x + y'\hat{a}_y + z'\hat{a}_z$$

$$x \to y \to z \to x$$

$$\bar{H} = \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{y'=-\frac{b}{2}}^{b/2} \int_{x'=-\frac{a}{2}}^{a/2} \frac{dx'dy'dz'\,\hat{a}_z \times \left(x\hat{a}_x - x'\hat{a}_x - y'\hat{a}_y - z'\hat{a}_z\right)}{|x\hat{a}_x - x'\hat{a}_x - y'\hat{a}_y - z'\hat{a}_z|^3}$$

$$\bar{H} = \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{y'=-\frac{b}{2}}^{b/2} \int_{x'=-\frac{a}{2}}^{a/2} \frac{(x-x')\hat{a}_y + y'\hat{a}_x + 0}{\left[(x-x')^2 + (-y')^2 + (-z')^2\right]^{3/2}}\, dx'dy'dz'$$

$\Rightarrow$ split into vector components + re-arrange order

$$\bar{H} = \hat{a}_x \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} \int_{y'=-\frac{b}{2}}^{b/2} \frac{y'dy'}{\left[y'^2 + (x-x')^2 + z'^2\right]^{3/2}} dx'dz'$$

$$+ \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-\frac{b}{2}}^{\frac{b}{2}} \int_{x'=-\frac{a}{2}}^{\frac{a}{2}} (x-x') \int_{z'=-\infty}^{\infty} \frac{dz'}{\left[(x-x')^2 + y'^2 + z'^2\right]^{3/2}} dx'dy'$$

$$= \hat{a}_x \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{x'=-\frac{a}{2}}^{a/2} \left[\frac{-1}{\sqrt{y'^2 + (x-x')^2 + z'^2}}\right]\Bigg|_{y'=-b/2}^{b/2} dx'dz'$$

$$+ \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \int_{x'=-\frac{a}{2}}^{a/2} (x-x') \left[\frac{\frac{z'}{(x-x')^2 + y'^2}}{\sqrt{(x-x')^2 + y'^2 + z'^2}}\right]\Bigg|_{z'=-\infty}^{\infty} dx'dy'$$

$$= \hat{a}_x \frac{I}{4\pi ab} \int_{z'=-\infty}^{\infty} \int_{x'=-a/2}^{a/2} \underbrace{\left[\frac{-1}{\sqrt{\frac{b^2}{4} + (x-x')^2 + z'^2}} + \frac{+1}{\sqrt{\frac{b^2}{4} + (x-x')^2 + z'^2}}\right]}_{\to 0} dx'dz'$$

0 (no $\hat{a}_x$ component)

$$+ \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \int_{x'=-a/2}^{a/2} \frac{x-x'}{(x-x')^2 + y'^2} \underbrace{\left[\frac{\infty}{\infty} - \frac{-\infty}{\infty}\right]}_{\to 2} dx'dy'$$

$$\bar{H} = \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \int_{x'=-a/2}^{a/2} \frac{2(x-x')\,dx'}{(x-x')^2 + y'^2}\, dy'$$

adapt $\int \frac{u\,du}{u^2+a^2}$

$$= \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \left[-\ln\left((x-x')^2 + y'^2\right) \right]\Bigg|_{x'=-a/2}^{a/2} dy'$$

$$= \hat{a}_y \frac{I}{4\pi ab} \int_{y'=-b/2}^{b/2} \left[-\ln\left((x-a/2)^2 + y'^2\right) + \ln\left((x+a/2)^2 + y'^2\right) \right] dy'$$

Next, use $\int \ln(u^2+a^2)\,du = u\ln(u^2+a^2) - 2u + 2a\tan^{-1}\frac{u}{a}$

$$\bar{H} = \hat{a}_y \frac{I}{4\pi ab} \Big[-y'\ln\left((x-a/2)^2 + y'^2\right) + \cancel{2y'} - 2(x-a/2)\tan^{-1}\left(\frac{y'}{x-a/2}\right)$$

cancel

$$+ y'\ln\left((x+a/2)^2 + y'^2\right) - \cancel{2y'} + 2(x+a/2)\tan^{-1}\left(\frac{y'}{x+a/2}\right) \Big]\Bigg|_{y'=-\frac{b}{2}}^{b/2}$$

$$= \hat{a}_y \frac{I}{4\pi ab} \Big[-\frac{b}{2}\ln\left((x-a/2)^2 + b^2/4\right) + \frac{-b}{2}\ln\left((x-\tfrac{a}{2})^2 + \frac{b^2}{4}\right)$$

$$-2(x-a/2)\tan^{-1}\left(\frac{b/2}{x-a/2}\right) + 2(x-a/2)\tan^{-1}\left(\frac{-b/2}{x-a/2}\right)$$

$$+ \frac{b}{2}\ln\left((x+a/2)^2 + b^2/4\right) + \frac{b}{2}\ln\left((x+a/2)^2 + b^2/4\right)$$

$$+ 2(x+a/2)\tan^{-1}\left(\frac{b/2}{x+a/2}\right) - 2(x+a/2)\tan^{-1}\left(\frac{-b/2}{x+a/2}\right) \Big]$$

Use Trigonometric identity $\tan^{-1}(-A) = -\tan^{-1}(A)$

$$\bar{H} = \hat{a}_y \frac{I}{4\pi ab}\left[-b\ln\left((x-a/2)^2+b^2/4\right)+b\ln\left((x+a/2)^2+b^2/4\right)\right.$$
$$-4(x-a/2)\tan^{-1}\left(\frac{b/2}{x-a/2}\right)$$
$$\left.+4(x+a/2)\tan^{-1}\left(\frac{b/2}{x+a/2}\right)\right]$$

$$\bar{H} = \hat{a}_y \frac{I}{4\pi ab}\left[b\ln\left(\frac{(x+a/2)^2+b^2/4}{(x-a/2)^2+b^2/4}\right)-4\left(x-\frac{a}{2}\right)\tan^{-1}\left(\frac{b/2}{x-a/2}\right)\right.$$
$$\left.+4(x+a/2)\tan^{-1}\left(\frac{b/2}{x+a/2}\right)\right]$$

valid for $y = z = 0$

Test Biot-Savart solution to long rectangular bar problem

Select arbitrary values for bar dimensions & current and location on *x*-axis.

$\mathbf{a} := 0.03\,\mathrm{m}$ $\mathbf{b} := 0.02\,\mathrm{m}$ $\mathbf{I} := 10\,\mathrm{A}$ $\mathbf{x} := 0.04\,\mathrm{m}$

Numerically solve integrals for H_x & H_y components of the magnetic field vector.

$$Hx\_num := \frac{I}{4\cdot\pi\cdot a\cdot b}\cdot\int_{-\infty}^{\infty}\int_{\frac{-a}{2}}^{\frac{a}{2}}\int_{\frac{-b}{2}}^{\frac{b}{2}}\frac{yp}{\left[(x-xp)^2+yp^2+zp^2\right]^{1.5}}\,dyp\,dxp\,dzp$$

$$Hy\_num := \frac{I}{4\cdot\pi\cdot a\cdot b}\cdot\int_{-\infty}^{\infty}\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-a}{2}}^{\frac{a}{2}}\frac{x-xp}{\left[(x-xp)^2+yp^2+zp^2\right]^{1.5}}\,dxp\,dyp\,dzp$$

Calculate analytic solutions H_x & H_y components of the magnetic field vector.

$Hxa := 0$

$$Hya := \frac{I}{4\cdot\pi\cdot a\cdot b}\cdot\left[b\cdot\ln\left[\frac{\left(x+\frac{a}{2}\right)^2+\frac{b^2}{4}}{\left(x-\frac{a}{2}\right)^2+\frac{b^2}{4}}\right]-4\cdot\left(x-\frac{a}{2}\right)\cdot\operatorname{atan}\left(\frac{\frac{b}{2}}{x-\frac{a}{2}}\right)\ldots\right.$$
$$\left.+4\cdot\left(x+\frac{a}{2}\right)\cdot\operatorname{atan}\left(\frac{\frac{b}{2}}{x+\frac{a}{2}}\right)\right]$$

Compare analytic and numeric solutions.

$\boxed{Hx\_num = 0}$ $\boxed{Hy\_num = 40.76686}$ A/m

Perfect Match!

$\boxed{Hxa = 0}$ $\boxed{Hya = 40.76686}$ A/m