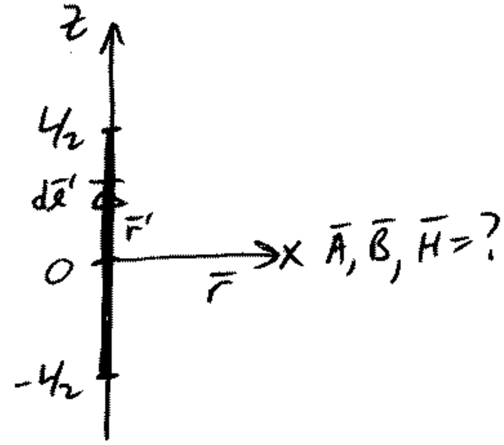
Example- Find the vector magnetic potential on the *x*-*y* plane for a straight wire segment of length L and carrying current I centered along the *z*-axis. Then, determine $\bar{B}$ and $\bar{H}$. Also, find $\bar{B}$ and $\bar{H}$ as $L \to \infty$.



$$\bar{A} = \int_{L'} \frac{\mu_o I \, d\bar{l}'}{4\pi |\bar{r} - \bar{r}'|}$$

$$d\bar{l}' = \hat{a}_z \, dz'$$

$$\bar{r}' = z' \hat{a}_z \quad -\frac{L}{2} < z' < \frac{L}{2}$$

$$\bar{r} = \rho \hat{a}_\rho$$

$$\bar{A} = \int_{z'=-L/2}^{L/2} \frac{\mu_o I \, \hat{a}_z \, dz'}{4\pi |\rho \hat{a}_\rho - z' \hat{a}_z|} = \hat{a}_z \frac{\mu_o I}{4\pi} \int_{z'=-L/2}^{L/2} \frac{dz'}{\sqrt{\rho^2 + z'^2}}$$

$$= \hat{a}_z \frac{\mu_o I}{4\pi} \ln\left(z' + \sqrt{z'^2 + \rho^2}\right) \Big|_{z'=-L/2}^{L/2}$$

$$\underline{\underline{\bar{A} = \hat{a}_z \frac{\mu_o I}{4\pi} \left(\ln\left(\frac{L}{2} + \sqrt{\left(\frac{L}{2}\right)^2 + \rho^2}\right) - \ln\left(-\frac{L}{2} + \sqrt{\left(\frac{L}{2}\right)^2 + \rho^2}\right) \right)}}$$

Check – Is $\bar{\nabla} \cdot \bar{A} = 0$? (Coulomb Gauge)

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \cancelto{0}{A_\rho}) + \frac{1}{\rho} \frac{\partial \cancelto{0}{A_\phi}}{\partial \phi} + \underbrace{\frac{\partial A_z}{\partial z}}_{\to 0} = 0 \quad \therefore \text{(OK)}$$

$$\bar{B} = \bar{\nabla}\times\bar{A} = \left[\frac{1}{\rho}\cancelto{0}{\frac{\partial A_z}{\partial\phi}} - \frac{\partial \cancelto{0}{A_\phi}}{\partial z}\right]\hat{a}_\rho + \left[\frac{\partial \cancelto{0}{A_\rho}}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\hat{a}_\phi$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho \cancelto{0}{A_\phi}\right) - \frac{\partial \cancelto{0}{A_\rho}}{\partial\phi}\right]\hat{a}_z$$

$$\bar{B} = -\frac{\partial A_z}{\partial\rho}\hat{a}_\phi = -\hat{a}_\phi \frac{\mu_0 I}{4\pi}\left[\frac{1}{L/2+\sqrt{(L/2)^2+\rho^2}}\left(\tfrac{1}{2}\right)\left[(L/2)^2+\rho^2\right]^{-1/2}(2\rho)\right.$$

$$\left. - \frac{1}{-L/2+\sqrt{(L/2)^2+\rho^2}}\left(\tfrac{1}{2}\right)\left[(L/2)^2+\rho^2\right]^{-1/2}(2\rho)\right]$$

$$= -\hat{a}_\phi \frac{\mu_0 I\rho}{4\pi\sqrt{(L/2)^2+\rho^2}}\left[\frac{1}{L/2+\sqrt{(L/2)^2+\rho^2}} - \frac{1}{-L/2+\sqrt{(L/2)^2+\rho^2}}\right]$$

$$= -\hat{a}_\phi \frac{\mu_0 I\rho}{4\pi\sqrt{(L/2)^2+\rho^2}}\left[\frac{\left(-L/2+\sqrt{(L/2)^2+\rho^2}\right) - \left(L/2+\sqrt{(L/2)^2+\rho^2}\right)}{\left(L/2+\sqrt{(L/2)^2+\rho^2}\right)\left(-L/2+\sqrt{(L/2)^2+\rho^2}\right)}\right]$$

$$= -\hat{a}_\phi \frac{\mu_0 I\rho}{4\pi\sqrt{(L/2)^2+\rho^2}}\left(\frac{-L}{-(L/2)^2+(L/2)^2+\rho^2}\right)$$

$$\bar{B} = +\hat{a}_\phi \frac{\mu_0 I}{4\pi\rho}\left(\frac{L}{\sqrt{(L/2)^2+\rho^2}}\right)$$

$$\bar{H} = \bar{B}/\mu_0 = \hat{a}_\phi \frac{I}{4\pi\rho}\left(\frac{L}{\sqrt{(L/2)^2+\rho^2}}\right)$$

What happens as $L \to \infty$ (infinitely long wire)?

$$\overline{B} = \lim_{L\to\infty} \frac{\mu_0 I}{4\pi\rho}\left(\frac{L}{\sqrt{(L/2)^2+\rho^2}}\right)\hat{a}_\phi$$

$$= \frac{\mu_0 I}{2\pi\rho} \lim_{L\to\infty}\left(\underbrace{\frac{L/2}{\sqrt{(L/2)^2+\rho^2}}}_{\to 1}\right)\hat{a}_\phi$$

$$\underline{\underline{\overline{B} = \hat{a}_\phi \frac{\mu_0 I}{2\pi\rho}}}$$

$$\underline{\underline{\overline{H} = \hat{a}_\phi \frac{I}{2\pi\rho}}}$$

Same as before using Biot-Savart & Ampere's Law!