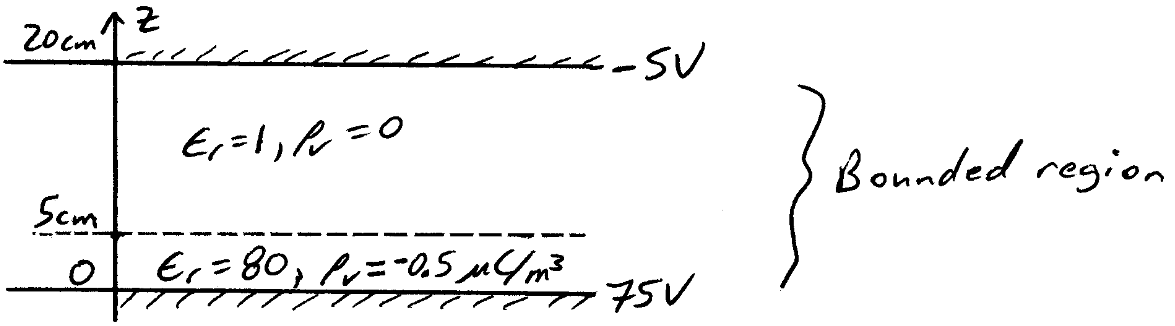
ex. The region between two large plates, located at $z=0$ and $z=20$ cm, is filled with a charged solution ($\epsilon_r = 80$, $\rho_v = -0.5\ \mu C/m^3$) from $0 \le z \le 5$ cm and air ($\epsilon_r = 1$) for $5 < z < 20$ cm. If the upper plate is held at -5 V and the lower plate at 75 V, find the potential, electric field, and electric flux density for $0 \le z \le 20$ cm.



For $\underline{0 \le z \le 5\text{cm}}$, $\rho_v \neq 0$. Therefore, Poisson's equation is appropriate. As the variation occurs in the z-direction, either the scalar Laplacian for rectangular or cylindrical coordinates may be used.

$$\nabla^2 V = \underbrace{\cancel{\frac{\partial^2 V}{\partial x^2}}}_{\to 0} + \underbrace{\cancel{\frac{\partial^2 V}{\partial y^2}}}_{\to 0} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_v}{\epsilon}$$

No variation

$$\int \left(\frac{d^2 V}{dz^2}\right) dz = \int \left(\frac{-\rho_v}{\epsilon}\right) dz$$

$$\frac{dV}{dz} = -\frac{\rho_v}{\epsilon} z + A_1$$

ex. cont.

$$\int \left(\frac{dV}{dz}\right) dz = \int \left[\frac{-\rho_v}{\epsilon} z + A_1\right] dz$$

$$V = -\frac{\rho_v}{2\epsilon} z^2 + A_1 z + B_1 = \frac{+0.5\times10^{-6} z^2}{2(80)(8.8542\times10^{-12})} + A_1 z + B_1$$

$$\underline{V = 352.93985\, z^2 + A_1 z + B_1 \qquad 0 \le z \le 5\,\text{cm}}$$

For $5 < z \le 20\,\text{cm}$, $\rho_v = 0$. Therefore, Laplace's equation is appropriate.

$$\nabla^2 V = \underbrace{\frac{\partial^2 V}{\partial x^2}}_{0} + \underbrace{\frac{\partial^2 V}{\partial y^2}}_{0} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\int \left(\frac{d^2 V}{dz^2}\right) dz = \int 0\, dz$$

$$\frac{dV}{dz} = A_2$$

$$\int \left(\frac{dV}{dz}\right) dz = \int A_2\, dz$$

$$\underline{V = A_2 z + B_2 \qquad 5 < z \le 20\,\text{cm}}$$

ex. cont.

Apply boundary conditions

① $V(z=0) = 75 = 352.9(0)^2 + A_1(0) + B_1$

$\hookrightarrow B_1 = 75$

$$\underline{V = 352.93985\, z^2 + A_1 z + 75 \text{ V} \quad 0 \le z \le 5\text{cm}}$$

② $V(z = 20\text{cm} = 0.2\text{m}) = -5 = A_2(0.2) + B_2$

$\hookrightarrow B_2 = -5 - 0.2 A_2$

$$V = A_2 z + (-5 - 0.2 A_2)$$

$$\underline{V = A_2(z - 0.2) - 5 \qquad 5 \le z \le 20\text{cm}}$$

③ For continuity, the voltage at $z = 5$cm must be the same for both expressions/equations

$$352.93985(0.05)^2 + A_1(0.05) + 75 = A_2(0.05 - 0.2) - 5$$

$$\hookrightarrow \underline{A_1(0.05) + A_2(0.15) = -80.88235}$$

$\Rightarrow$ Still 2 coefficients w/ only 1 equation $\Leftarrow$

④ Use $D_{1n} - D_{2n} = \rho_s = 0 \Rightarrow D_{1n} = D_{2n}$

$$\bar{D} = \epsilon \bar{E} = \epsilon(-\bar{\nabla} V) = -\epsilon\left[\hat{a}_x \underbrace{\cancel{\frac{\partial V}{\partial x}}}_{\to 0} + \hat{a}_y \underbrace{\cancel{\frac{\partial V}{\partial y}}}_{\to 0} + \hat{a}_z \frac{\partial V}{\partial z}\right]$$

$$= -\hat{a}_z \epsilon \frac{\partial V}{\partial z}$$

ex. cont.

④ cont.

$$\bar{D} = \begin{cases} -\hat{a}_z\, 80\epsilon_0 \frac{\partial}{\partial z}\left(352.93985\, z^2 + A_1 z + 75\right) & 0 \le z \le 5\text{cm} \\ -\hat{a}_z\, \epsilon_0 \frac{\partial}{\partial z}\left(A_2 z - 0.2 A_2 - 5\right) & 5 \le z \le 20\text{cm} \end{cases}$$

$$= \begin{cases} -\hat{a}_z\left(5\times10^{-7} z + 80\epsilon_0 A_1\right) & 0 \le z \le 5\text{cm} \\ -\hat{a}_z\, A_2 \epsilon_0 & 5 \le z \le 20\text{cm} \end{cases}$$

At $z = 5\text{cm}$

$$-\left(5\times10^{-7}(0.05) + 80\epsilon_0 A_1\right) = -A_2\epsilon_0$$

$$\hookrightarrow \quad \underline{-80A_1 + A_2 = \frac{5\times10^{-7}(0.05)}{\epsilon_0} = 2823.51878}$$

$\Rightarrow$ Between ③ + ④, we now have 2 eqns − 2 unknowns.

Using my TI-68,

$$A_1 = -41.859765 \quad + \quad A_2 = -525.2624$$

$$\boxed{V = \begin{cases} 352.93985\, z^2 - 41.859765\, z + 75\text{ V} & 0 \le z \le 5\text{cm} \\ -525.2624\, z + 100.0525\text{ V} & 5\text{cm} \le z \le 20\text{cm} \end{cases}}$$

Check ① $V(0) = \underline{75\text{ V}}$ (ok)

② $V(20\text{cm}) = -525.2624(0.2) + 100.0525 = \underline{-5\text{V}}$ (ok)

③ $V(5\text{cm}) = 352.94(0.05)^2 - 41.86(0.05) + 75 \stackrel{?}{=} -525.26(0.05) + 100.05$

$\underline{V(5\text{cm}) = 73.789 = 73.789}$ ok ex cont.

check cont. (4) $D_{1n} = D_{2n}$ @ $z = 0.05\,m$

$$-\left(5\times10^{-7}(0.05) + 80\epsilon_0(-41.86)\right) \overset{?}{\doteq} -(-525.2624)\epsilon_0$$

$$\underline{4.651\times10^{-9} = 4.651\times10^{-9}} \quad \text{(ok)}$$

$$\bar{E} = -\bar{\nabla}V = -\hat{a}_x \underbrace{\frac{\partial V}{\partial x}}_{\to 0} - \hat{a}_y \underbrace{\frac{\partial V}{\partial y}}_{\to 0} - \hat{a}_z \frac{\partial V}{\partial z}$$

$$= \begin{cases} -\hat{a}_z \frac{d}{dz}\left(352.94z^2 - 41.86z + 75\right) & 0 \le z \le 5cm \\ -\hat{a}_z \frac{d}{dz}\left(-525.2624z + 100.0525\right) & 5 \le z \le 20cm \end{cases}$$

$$\boxed{\bar{E} = \begin{cases} \hat{a}_z(-705.88z + 41.86)\ V/m & 0 \le z \le 5cm \\ \hat{a}_z\ 525.2624\ V/m & 5 \le z \le 20cm \end{cases}}$$

$$\bar{D} = \epsilon\bar{E} = \begin{cases} 80\epsilon_0\hat{a}_z(-705.88z + 41.86) & 0 \le z \le 5cm \\ \epsilon_0\hat{a}_z\,525.2624 & 5 \le z \le 20cm \end{cases}$$

$$\boxed{\bar{D} = \begin{cases} \hat{a}_z\left(-5\times10^{-7}z + 2.965\times10^{-8}\right) C/m^2 & 0 \le z \le 5cm \\ \hat{a}_z\,4.6508\times10^{-9}\ C/m^2 & 5 \le z \le 20cm \end{cases}}$$