Magnetic Circuits

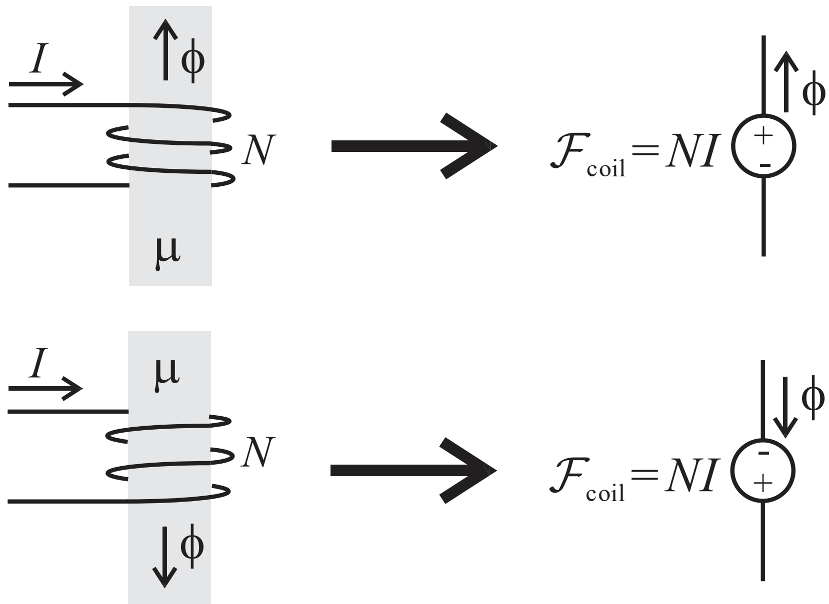
- This gives approximate results (~ ±5%). To get exact results, either involves solving very difficult integral equations or getting numerical solutions from electromagnetic (EM) modeling software.
- Assume all magnetic flux in core & magnetic permeability μ is constant for each material.
- Assume uniform magnetic flux density B distribution in core (not accurate near corners).
- Assume magnetic flux follows mean path/length ℓ through core and that sections of core have uniform cross-sectional areas.

Analogies between Electric and Magnetic Circuits

Electric	**Magnetic**
Conductivity σ (S/m)	Magnetic permeability μ (H/m)
Electric field intensity E (V/m)	Magnetic field intensity H (A/m)
Electric current $I = \iint_s \overline{J} \bullet d\overline{s}$ (A)	Magnetic flux $\Psi = \iint_S \overline{B} \bullet d\overline{s} = \lvert\overline{B}\rvert S$ (uniform) or, EE 330, $\phi = \iint_A \overline{B} \bullet d\overline{A} = \lvert\overline{B}\rvert A = BA$
Electric current density $J = I/S = \sigma E$ (A/m^2) or, EE 330, $J = I/A = \sigma E$ (A/m^2)	Magnetic flux density $B = \Psi/S = \mu H$ (Wb/m^2) or, EE 330, $B = \phi/A = \mu H$ (Wb/m^2)
Electromotive force (emf) or voltage V (V)	Magnetomotive force (mmf) $\mathcal{F}$(A-turns or A·t)
Resistance R (Ω)	Reluctance $\mathcal{R}$ (A·t/Wb)
Conductance $G = 1/R$ (S)	Permeance $\mathcal{P} = 1/\mathcal{R}$ (Wb/A·t)
Ohm's Law $V = IR = E\ell \Rightarrow R = V/I = \ell/\sigma S$ or, EE 330, $R = V/I = \ell/\sigma A$	'Ohm's Law' $\mathcal{F} = \Psi\mathcal{R} = H\ell \Rightarrow \mathcal{R} = \mathcal{F}/\Psi = \ell/\mu S$ or, EE 330, $\mathcal{F} = \phi\mathcal{R} = H\ell \Rightarrow \mathcal{R} = \mathcal{F}/\phi = \ell/\mu A$
Kirchoff's laws: $\Sigma I = 0$ (KCL) $\Sigma V - \Sigma IR = 0$ (KVL)	'Kirchoff's laws': $\Sigma\Psi = \Sigma\phi = 0$ (KCL) $\Sigma\mathcal{F} - \Sigma\Psi\mathcal{R} = \Sigma\mathcal{F} - \Sigma\phi\mathcal{R} = 0$ (KVL)

* Based on Table 8.4 of *Elements of Electromagnetics* (Seventh Edition), Sadiku, Oxford, 2018.

- mmf ***source*** coils $\underline{\boldsymbol{\mathcal{F}_{\text{source}} = NI = Ni}}$ (analogous to voltage sources). Polarity is determined by right hand rule (RHR), i.e., curl fingers of right hand in direction of positive current flow on coil around core, thumb points toward direction of magnetic flux ϕ and '+' terminal of mmf source.



- Coil inductance $\underline{L = \frac{\lambda}{i} = \frac{N\phi}{i} = \frac{N^2}{\mathcal{R}}}$ (H) where N is number of turns of the coil and $\mathcal{R}$ is equivalent reluctance 'seen' by coil.

- With Kirchoff's Laws analogies, series & parallel reluctance, mmf division, and flux division rules work:

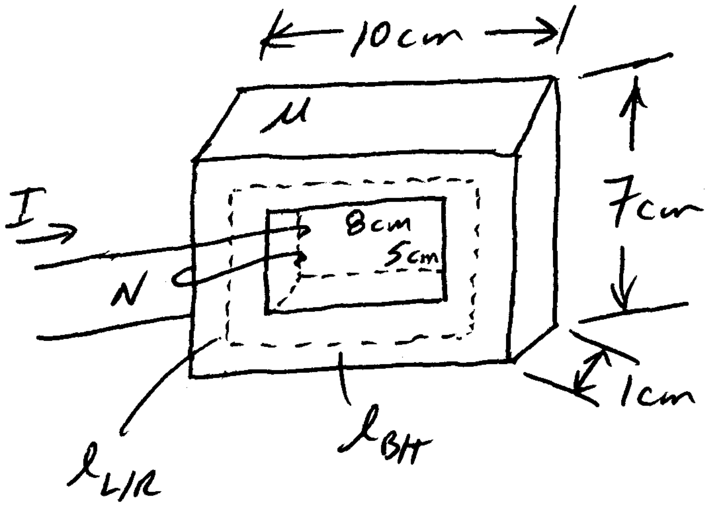
  Series reluctance: $\mathcal{R}_{\text{eq}} = \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \ldots$

  Parallel reluctance: $1/\mathcal{R}_{\text{eq}} = 1/\mathcal{R}_1 + 1/\mathcal{R}_2 + 1/\mathcal{R}_3 + \ldots$

  mmf division for series reluctances: $\mathcal{F}_i = \mathcal{F}_{\text{applied}}\,(\mathcal{R}_i / \mathcal{R}_{\text{eq}})$

  flux division for parallel reluctances: $\phi_i = \phi_{\text{in}}\,(\mathcal{R}_{\text{eq}} / \mathcal{R}_i)$

Example- For the problem geometry shown, determine the equivalent magnetic circuit. Then, use the magnetic circuit to find the magnetic flux, magnetic flux density, and magnetic field in the core as well as the inductance of the coil, energy stored in the magnetic field, and mmf drops on each side of the core.



$N = 50$ turns

$I = 4A$

$\mu = 4000\mu_0$

$$l_{L/R} = 7\,cm - \left(\frac{7-5}{2}\right) = 7 - 1 = 6\,cm = \underline{0.06\,m}$$

$$l_{B/T} = 10\,cm - \left(\frac{10-8}{2}\right) = 10 - 1 = 9\,cm = \underline{0.09\,m}$$

$$A = 1\,cm\,(1\,cm) = \underline{10^{-4}\,m^2}$$

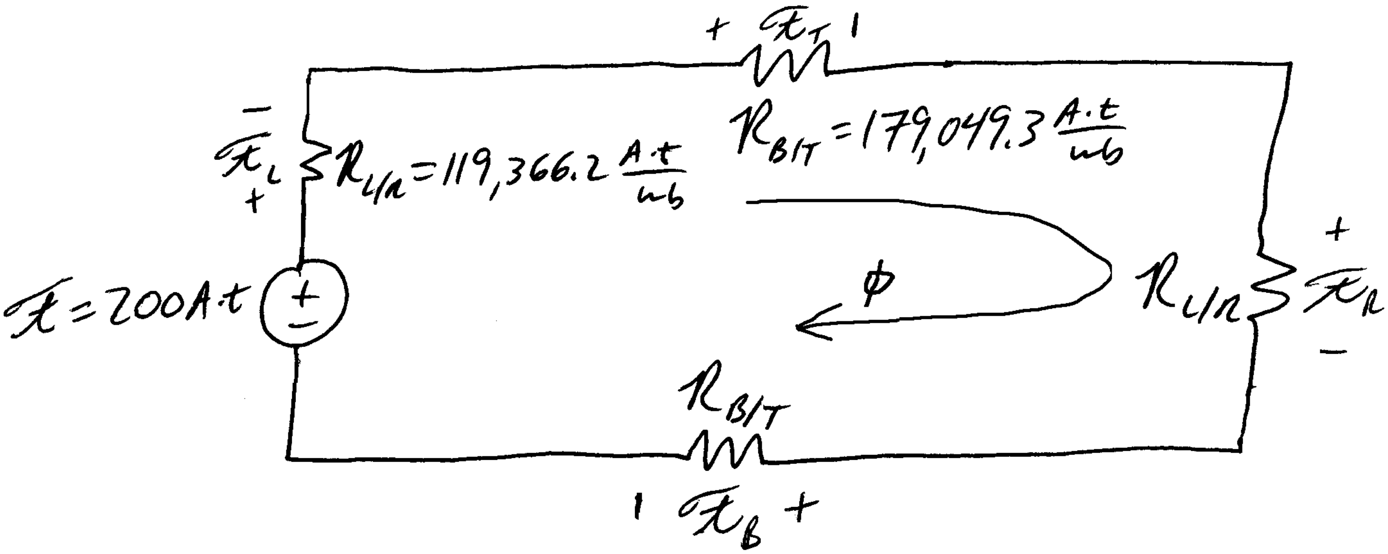
$$\mathcal{R}_{L/R} = \frac{l_{L/R}}{\mu A} = \frac{0.06}{4000(4\pi \times 10^{-7})(10^{-4})} = \underline{119{,}366.2\ \frac{A\cdot t}{wb}}$$

$$\mathcal{R}_{B/T} = \frac{l_{B/T}}{\mu A} = \frac{0.09}{4000(4\pi \times 10^{-7})(10^{-4})} = \underline{179{,}049.3\ \frac{A\cdot t}{wb}}$$

mmf source $\mathcal{F} = NI = 50(4) = \underline{200\ A\cdot t}$

ex. cont.

Equivalent Magnetic Circuit



$$\mathcal{R}_{eq} = \mathcal{R}_{L/R} + \mathcal{R}_{B/T} + \mathcal{R}_{L/R} + \mathcal{R}_{B/T}$$

$$= 2(119{,}366.2) + 2(179{,}049.3)$$

$$\underline{\mathcal{R}_{eq} = 596{,}831 \ A \cdot t / wb}$$

$$\phi = \frac{\mathcal{F}}{\mathcal{R}_{eq}} = \frac{200}{596{,}831} = \underline{\underline{0.3351 \ mWb}}$$

$$L = N^2 / \mathcal{R}_{eq} = \frac{50^2}{596{,}831} = \underline{\underline{4.189 \ mH}}$$

$$W_m = \tfrac{1}{2} L I^2 = \tfrac{1}{2}(4.189 \times 10^{-3})(4^2) = \underline{\underline{33.51 \ mJ}}$$

ex. cont.

$$B = |\bar{B}| = \frac{\psi}{A} = \frac{3.351\times10^{-4}}{10^{-4}} = \underline{\underline{3.351 \text{ Wb/m}^2}}$$

$$|\bar{H}| = H = B/\mu = \frac{3.351}{4000(4\pi\times10^{-7})} = \underline{\underline{666.66 \text{ A/m}}}$$

mmf drops on each side of core?

$$\text{mmf}_{B/T} = \mathcal{F}_{B/T} = \Phi\,\mathcal{R}_{B/T} = (3.351\times10^{-4})(179{,}049.3)$$

$$\underline{\underline{\mathcal{F}_{B/T} = 60 \text{ A}\cdot\text{t}}}$$

"KVL"

$$\mathcal{F} = \mathcal{F}_B + \mathcal{F}_T + \mathcal{F}_L + \mathcal{F}_R$$

$$\hookrightarrow \mathcal{F}_L + \mathcal{F}_R = 200 - 60 - 60 = 80$$

$$\mathcal{F}_{L/R} = 80/2 = \underline{\underline{40 \text{ A}\cdot\text{t}}}$$

Alt. way to get H, $\mathcal{F} = H\ell$

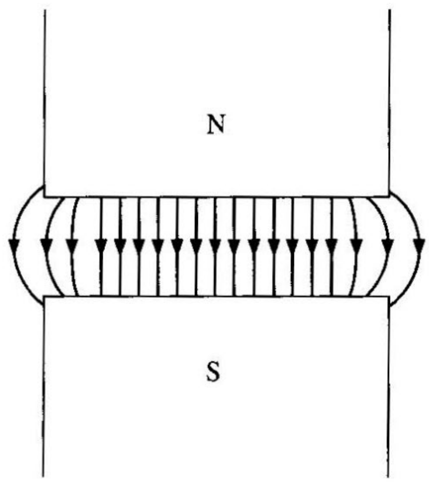
$$\mathcal{F}_{B/T} = 60 \text{ A}\cdot\text{t} = H(0.09\text{m})$$

$$\hookrightarrow \underline{H = \frac{60}{0.09} = 666.\bar{6} \text{ A/m}} \quad \text{Same!}$$

Special case- air gaps

- Small air gaps occur (and are necessary) in motors, generators, …
- Because the magnetic permeability of air $\mu_{\text{air}} \approx \mu_0$, these air gaps have large reluctances $\mathcal{R}_{\text{gap}} = \ell_{\text{gap}} / \mu_0 A_{\text{gap}}$ and a disproportionate effect on the magnetic circuit.
- Further, as shown in Figure 1-6, the magnetic field, flux density, and flux experience fringing at air gaps, i.e., they tend to 'spread out'.

Figure 1–6 | The fringing effect of a magnetic field at an air gap. Note the increased cross-sectional area of the air gap compared with the cross-sectional area of the metal.

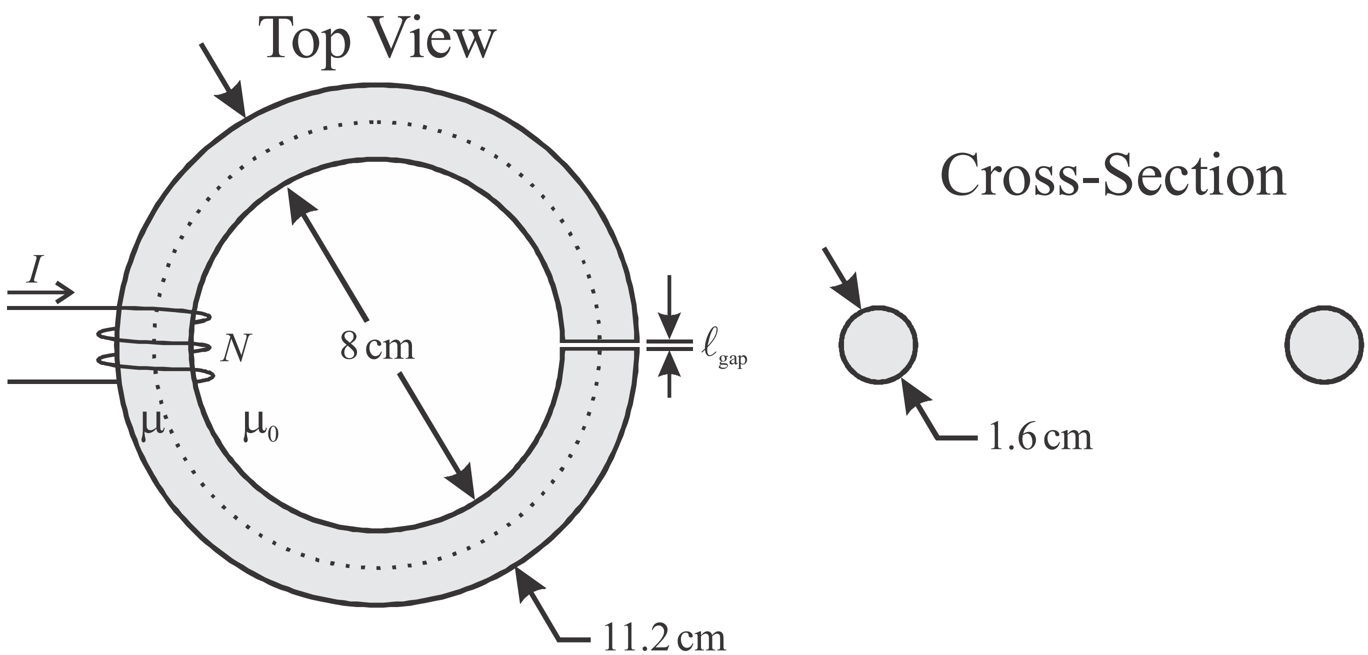


- The fringing effect can be partially accounted for by adjusting/correcting A_{gap} to be larger than the cross-sectional area of the core, i.e., $A_{\text{gap}} > A_{\text{core}}$. The adjustment amount (e.g., 5% increase or $A_{\text{gap}} = 1.05A_{\text{core}}$) can be given by a graph or table based on institutional knowledge or EM modeling software.
- Alternatively, a rough 'rule of thumb', when ℓ_{gap} is much less than the transverse dimension(s) of the core, is to add ℓ_{gap} to the applicable transverse dimensions when calculating A_{gap}.

E.g., For a core with circular cross-section of radius a, the core cross-sectional area is computed as $A_{\text{core}} = \pi a^2$. For the air gap, we adjust the radius to $a_{\text{gap}} = a + \ell_{\text{gap}}$. Then, compute the adjusted gap area as $\underline{\boldsymbol{A_{\text{gap}} = \pi(a + \ell_{\text{gap}})^2}}$.

E.g., For a core with a rectangular cross-section, $a \times b$, $A_{\text{core}} = a\,b$. For the air gap, we adjust the dimensions to $a_{\text{gap}} = a + \ell_{\text{gap}}$ and $b_{\text{gap}} = b + \ell_{\text{gap}}$ Then, compute the adjusted gap area as $\underline{\boldsymbol{A_{\text{gap}} = (a + \ell_{\text{gap}})\,(b + \ell_{\text{gap}})}}$.

Example- One hundred turns of wire carrying 0.5 A of current is wrapped around a toroid with an inner diameter of 8 cm and outer diameter of 11.2 cm and having circular cross-section. The toroid was made by bending a cobalt ($\mu = 250\mu_0$) rod around a circular form, leaving a 1.25 mm gap. Use magnetic circuits to determine the various magnetic quantities related to this problem first neglecting fringing and then estimating the effects of fringing at the gap.



No fringing (i.e., $A_{\text{gap}} = A_{\text{core}}$)

$$\mathcal{R}_{\text{core}} = \frac{\ell_{\text{core}}}{\mu_{\text{core}} A_{\text{core}}} = \frac{\pi d_{\text{ave}} - \ell_{\text{gap}}}{\mu_{\text{core}} (\pi a_{\text{core}}^2)} = \frac{\pi(9.6 \cdot 10^{-2}) - 1.25 \cdot 10^{-3}}{250(4\pi \cdot 10^{-7})\,[\pi(0.8 \cdot 10^{-2})^2]}$$

$$= \frac{0.300342895}{250(4\pi \cdot 10^{-7})0.000201062} = \underline{4{,}754{,}857.3 \ \text{A} \cdot \text{t}/\text{Wb}}$$

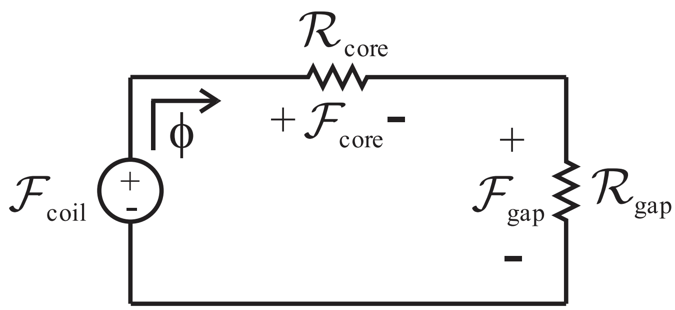
$$\mathcal{R}_{\text{gap}} = \frac{\ell_{\text{gap}}}{\mu_0 A_{\text{gap}}} = \frac{\ell_{\text{gap}}}{\mu_0 A_{\text{core}}} = \frac{1.25 \cdot 10^{-3}}{4\pi \cdot 10^{-7}\,[0.000201062]}$$

$$= \underline{4{,}947{,}321.7 \ \text{A} \cdot \text{t}/\text{Wb}}$$

Note how $\mathcal{R}_{\text{gap}} > \mathcal{R}_{\text{core}}$ despite $\ell_{\text{gap}} << \ell_{\text{core}}$.

$$\mathcal{F}_{\text{coil}} = NI = 100(0.5) = \underline{50 \ \text{A} \cdot \text{t}}$$

Magnetic Circuit



$$\phi=\frac{\mathcal{F}_{\text{coil}}}{\mathcal{R}_{\text{core}}+\mathcal{R}_{\text{gap}}}=\frac{50}{4,754,857.3+4,947,321.7}=5.15348\cdot 10^{-6}\ \text{Wb}=\underline{5.1535\ \mu\text{Wb}}$$

$$B_{\text{gap}}=B_{\text{core}}=\frac{\phi}{A_{\text{core}}}=\frac{5.15348\cdot 10^{-6}}{0.000201062}=0.0256313\ {}^{\text{Wb}}\!/_{\text{m}^2}=\underline{25.631\,\text{mT}}$$

$$\mathcal{F}_{\text{core}}=\phi\mathcal{R}_{\text{core}}=5.15348\cdot 10^{-6}(4,754,857.3)=\underline{24.504\ \text{A}\cdot\text{t}}\qquad \text{OR}$$

$$=\mathcal{F}_{\text{coil}}\frac{\mathcal{R}_{\text{core}}}{\mathcal{R}_{\text{core}}+\mathcal{R}_{\text{gap}}}=50\frac{4,754,857.3}{4,754,857.3+4,947,321.7}=\underline{24.504\ \text{A}\cdot\text{t}}$$

$$\mathcal{F}_{\text{gap}}=\phi\mathcal{R}_{\text{gap}}=5.15348\cdot 10^{-6}(4,947,321.7)=\underline{25.496\ \text{A}\cdot\text{t}}\qquad \text{OR}$$

$$=\mathcal{F}_{\text{coil}}-\mathcal{F}_{\text{core}}=50-24.504=\underline{25.496\ \text{A}\cdot\text{t}}$$

$$H_{\text{core}}=\frac{B_{\text{core}}}{\mu_{\text{core}}}=\frac{0.0256313}{250(4\pi\bullet 10^{-7})}=\underline{81.587\ {}^{\text{A}}\!/_{\text{m}}}\qquad \text{OR}$$

$$=\frac{\mathcal{F}_{\text{core}}}{\ell_{\text{core}}}=\frac{24.504}{0.300342895}=\underline{81.587\ {}^{\text{A}}\!/_{\text{m}}}$$

$$H_{\text{gap}}=\frac{B_{\text{gap}}}{\mu_0}=\frac{0.0256313}{4\pi\bullet 10^{-7}}=\underline{20,397\ {}^{\text{A}}\!/_{\text{m}}}=\frac{\mathcal{F}_{\text{gap}}}{\ell_{\text{gap}}}=\frac{25.496}{1.25\bullet 10^{-3}}=\underline{20,397\ {}^{\text{A}}\!/_{\text{m}}}$$

$$L_{\text{coil}}=\frac{N\phi}{I}=\frac{100(5.15348\cdot 10^{-6})}{0.5}=0.0010307\,\text{H}=\underline{1.0307\ \text{mH}}\quad \text{OR}$$

$$=\frac{N^2}{\mathcal{R}_{\text{eq}}}=\frac{N^2}{\mathcal{R}_{\text{core}}+\mathcal{R}_{\text{gap}}}=\frac{100^2}{4,754,857.3+4,947,321.7}=\underline{1.0307\ \text{mH}}$$

<u>w/ fringing</u> (i.e., $A_{\text{gap}} > A_{\text{core}}$)

$$\mathcal{R}_{\text{core}} = \underline{4{,}754{,}857.3 \ {}^{\text{A}\cdot\text{t}}\!/\!_{\text{Wb}}} \quad \text{and} \quad \mathcal{F}_{\text{coil}} = \underline{50 \ \text{A}\cdot\text{t}} \quad \text{(unchanged)}$$

Use 'rule of thumb' to account for fringing in air gap-

$$\mathcal{R}_{\text{gap}} = \frac{\ell_{\text{gap}}}{\mu_0 A_{\text{gap}}} = \frac{\ell_{\text{gap}}}{\mu_0 \pi (a_{\text{core}} + \ell_{\text{gap}})^2} = \frac{1.25 \bullet 10^{-3}}{4\pi \bullet 10^{-7} [\pi (0.8 \bullet 10^{-2} + 1.25 \bullet 10^{-3})^2]}$$

$$= \frac{1.25 \bullet 10^{-3}}{4\pi \bullet 10^{-7} (0.000268803)} = \underline{3{,}700{,}548 \ {}^{\text{A}\cdot\text{t}}\!/\!_{\text{Wb}}}$$

Note: $\dfrac{A_{\text{gap}}}{A_{\text{core}}} = \dfrac{0.000268803}{0.000201062} = 1.337$ and now $\mathcal{R}_{\text{gap}} < \mathcal{R}_{\text{core}}$.

$$\phi = \frac{\mathcal{F}_{\text{coil}}}{\mathcal{R}_{\text{core}} + \mathcal{R}_{\text{gap}}} = \frac{50}{4{,}754{,}857.3 + 3{,}700{,}548} = 5.913377 \cdot 10^{-6} \ \text{Wb} = \underline{5.9134 \ \mu\text{Wb}}$$

$$B_{\text{core}} = \frac{\phi}{A_{\text{core}}} = \frac{5.913377 \cdot 10^{-6}}{0.000201062} = 0.0294107 \ {}^{\text{Wb}}\!/\!_{\text{m}^2} = \underline{29.4107 \ \text{mT}}$$

$$B_{\text{gap}} = \frac{\phi}{A_{\text{gap}}} = \frac{5.913377 \cdot 10^{-6}}{0.000268803} = 0.0219989 \ {}^{\text{Wb}}\!/\!_{\text{m}^2} = \underline{21.9989 \ \text{mT}}$$

$$\mathcal{F}_{\text{core}} = \phi \mathcal{R}_{\text{core}} = 5.913377 \cdot 10^{-6} (4{,}754{,}857.3) = \underline{28.117 \ \text{A}\cdot\text{t}}$$

$$\mathcal{F}_{\text{gap}} = \phi \mathcal{R}_{\text{gap}} = 5.913377 \cdot 10^{-6} (3{,}700{,}548210021.5) = \underline{21.883 \ \text{A}\cdot\text{t}}$$

$$H_{\text{core}} = \frac{B_{\text{core}}}{\mu_{\text{core}}} = \frac{0.0294107}{250(4\pi \bullet 10^{-7})} = \underline{93.617 \ {}^{\text{A}}\!/\!_{\text{m}}}$$

$$H_{\text{gap}} = \frac{B_{\text{gap}}}{\mu_0} = \frac{0.0219989}{4\pi \bullet 10^{-7}} = \underline{17{,}506.2 \ {}^{\text{A}}\!/\!_{\text{m}}}$$

$$L_{\text{coil}} = \frac{N\phi}{I} = \frac{100(5.913377 \cdot 10^{-6})}{0.5} = 0.001182675 \, \text{H} = \underline{1.183 \ \text{mH}}$$

Accounting for fringing changed values significantly, e.g., ~15% for L_{coil}.