Examine using the FFT to approximate the continuous-time Fourier transform (CTFT) of $x(t) = 4e^{-4t}\cos(10t)\,u(t)$.

a) Find the CTFT of $x(t)$.

b) Using Matlab, create a sampled version $x[n]$ of $x(t)$ where $t_n = nT$. For each case below, plot $x(t)$ (solid line) and $x[n]$ (dots) for $0 \le t \le (N\text{-}1)T$ and vertical scale of -1.5 to 4. On the same page, plot $|X(\omega)|$ (solid line) and the FFT approximation (use contfft() function) to the CTFT (dots) for $0 \le \omega \le 60$ rad/s and vertical scale of 0 to 0.7. Compute and list $\Gamma = \Delta\omega_k$ and $\max(\omega_k)$. Cases: (i) $T = 0.04$ s and $N = 32$, (ii) $T = 0.02$ s and $N = 64$, (iii) $T = 0.02$ s and $N = 128$, and (iv) $T = 0.01$ s and $N = 512$.

c) Comment on how the accuracy and resolution of the FFT approximation to the CTFT change with respect to sampling rate and number of data points.

From Table 3.2, $e^{-bt}u(t) \leftrightarrow \dfrac{1}{j\omega + b}$ $\quad b > 0.$

From Table 3.1,

linearity $\quad a\,x(t) \leftrightarrow a\,X(\omega)$

mult. by $\cos(\omega_0 t)$ $\quad x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$

So, $\quad 4e^{-4t}u(t) \leftrightarrow 4\dfrac{1}{j\omega+4}$

and $\quad 4e^{-4t}u(t)\cos(10t) \leftrightarrow \frac{1}{2}\left[\dfrac{4}{j(\omega+10)+4} + \dfrac{4}{j(\omega-10)+4}\right]$

yielding

$$X(\omega) = \frac{2}{j(\omega+10)+4} + \frac{2}{j(\omega-10)+4} \qquad -\infty < \omega < \infty$$

b)

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% Chapter 4 problem 4.19 (chap_CTFT_w_FFT_A.m)
% EE 313, your name, mm-dd-yyyy
% Use FFT to approximate the Fourier transform of x(t) = 4e^-4t u(t)
% at discrete frequencies wk = 2*pi*k/N/T and compare with the exact
% answer X[w] = 4/(jw+4) over frequency range 0 < w < 2*pi*(N-1)/N/T.
% This m-file does FFT approx. of X(w)using contfft().
clear; clc; close all;
N = input('Input number N of sample points = ');
T = input('Input sample rate T (s) = ');
Gamma = 2*pi/N/T;
t = 0 : 0.01 : (N-1)*T; x = 4*exp(-4*t).*cos(10*t); % exact x(t)
tn = 0 : T : (N-1)*T; xn = 4*exp(-4*tn).*cos(10*tn); % sampled x(t)
w = 0 : 0.2 : 60; % frequency range for Fourier transform
X = 2./(j*(w+10)+4)+ 2./(j*(w-10)+4); % exact expression for CTFT
% Fourier transform of xn[n]=x(nT) & approx. to X(w)
[Xfft,wk] = contfft(xn,T);
wkmax=max(wk)
Xmagfft = abs(Xfft);  % FFT FT approx. spectrum
Xmag = abs(X);        % exact FT spectrum
% Plot input signal
plot(t,x,'r-',tn,xn,'b.'), axis([0 (N-1)*T -1.5 4]),
ylabel('{\itx}({\itt})','fontsize',14,'fontname','times'),
xlabel('{\itt} (sec)','fontsize',14,'fontname','times'),
title({['Input signal w/ {\itT} = ',num2str(T),' s and {\itN} = ',...
    num2str(N)]},'fontsize',16,'fontname','times'),
legend(' continuous {\itx}({\itt}) =  4e^{-4{\itt} }{\itu}({\itt})',...
    ' sampled {\itx}({\itnT})'),
% Plot Fourier transform of signal
figure,plot(w,Xmag,'r-',wk,Xmagfft,'b.'),
axis([0 60 0 0.7]),
ylabel('|{\itX}(\omega)|','fontsize',14,'fontname','times'),
xlabel('\omega (rad/sec)','fontsize',14,'fontname','times'),
title({['FT of {\itx}({\itt}) w/ {\itT} = ',num2str(T),' s, {\itN} =
',num2str(N),...
    ', \Gamma = ',num2str(Gamma,2),' & \omega_{k,max }= ',num2str(wkmax,4),...
    ' rad/s']},'fontsize',13,'fontname','times'),
legend('|{\itX}(\omega)| exact','|{\itX}(\omega)| FFT using contfft.m'),
set(findobj('type','line'),'linewidth',1.5,'markersize',14)
set(findobj('type','axes'),'linewidth',2,'fontname','times')
set(findobj('text','line'),'fontname','times')
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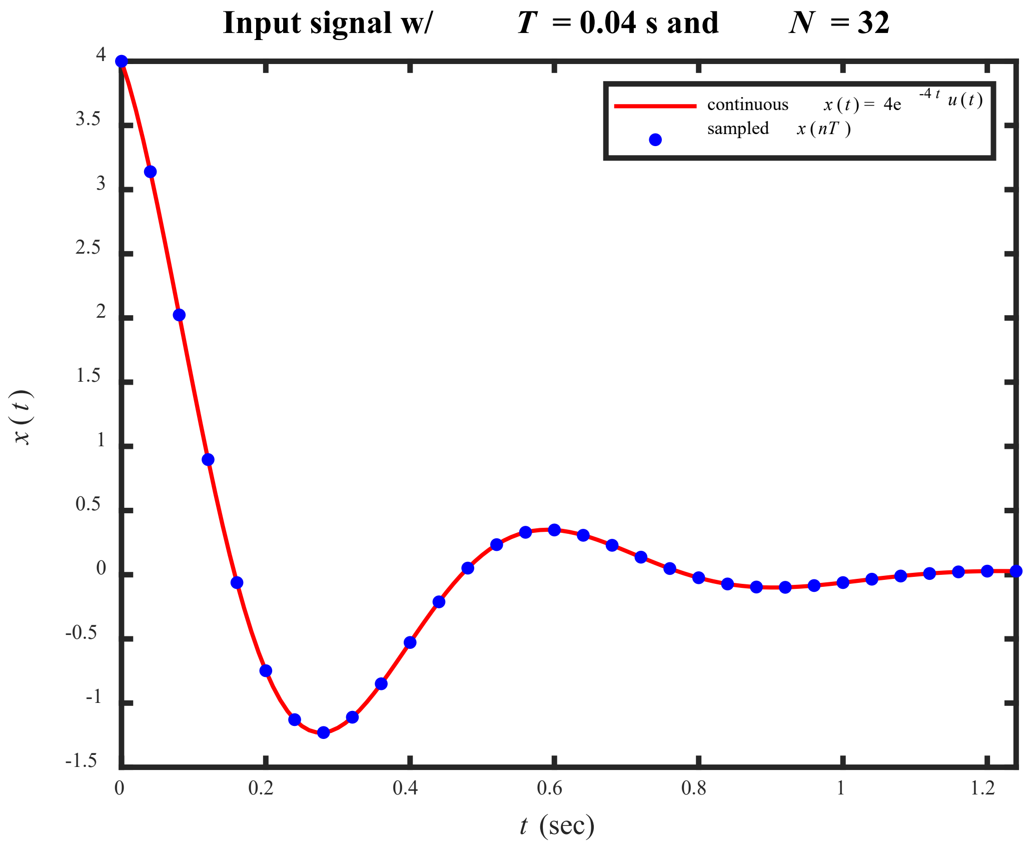
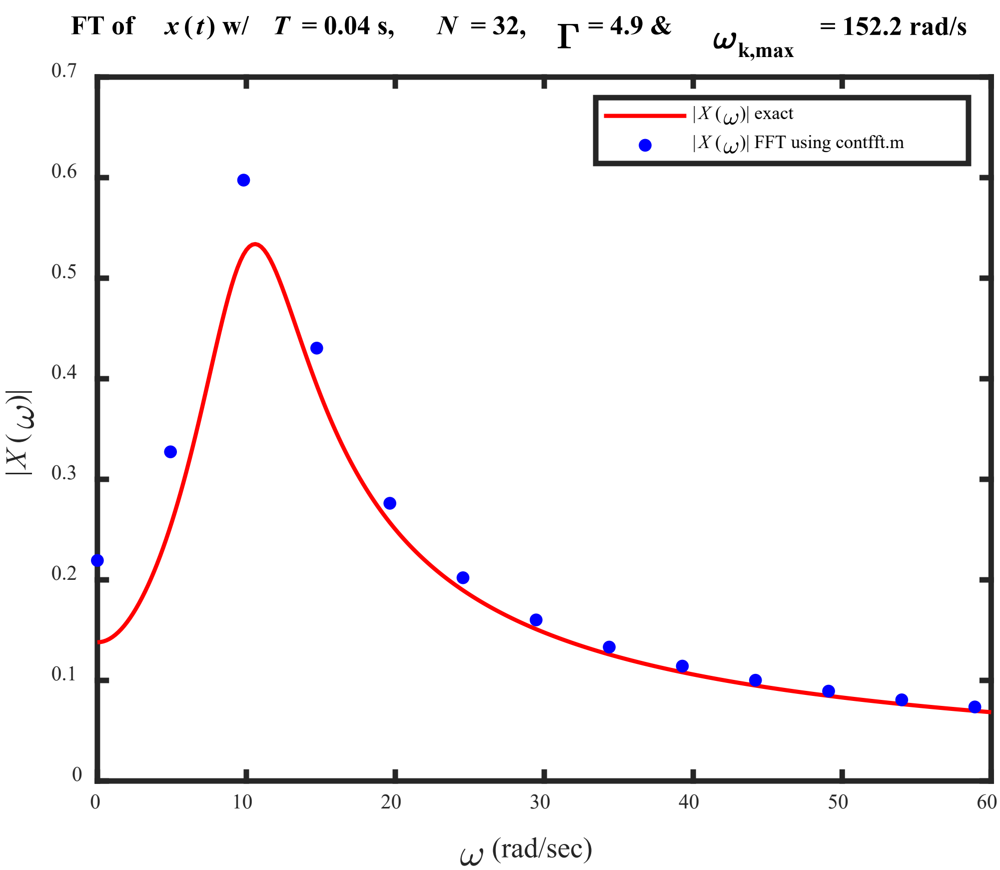
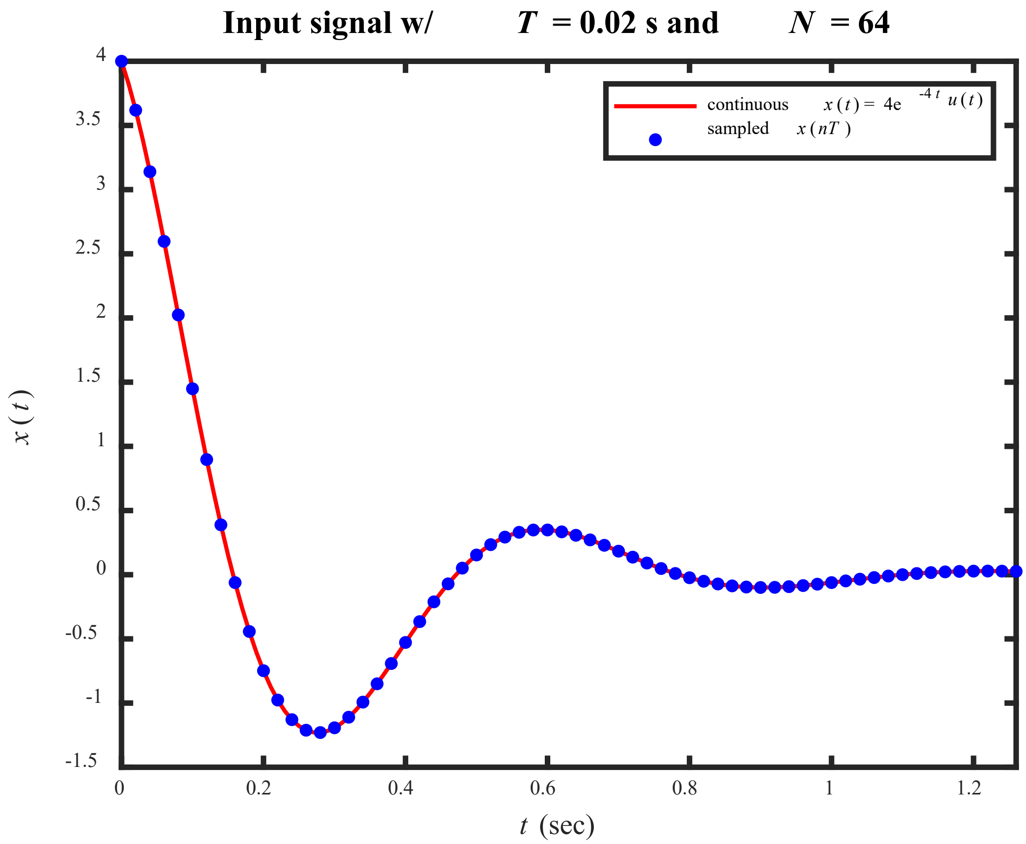
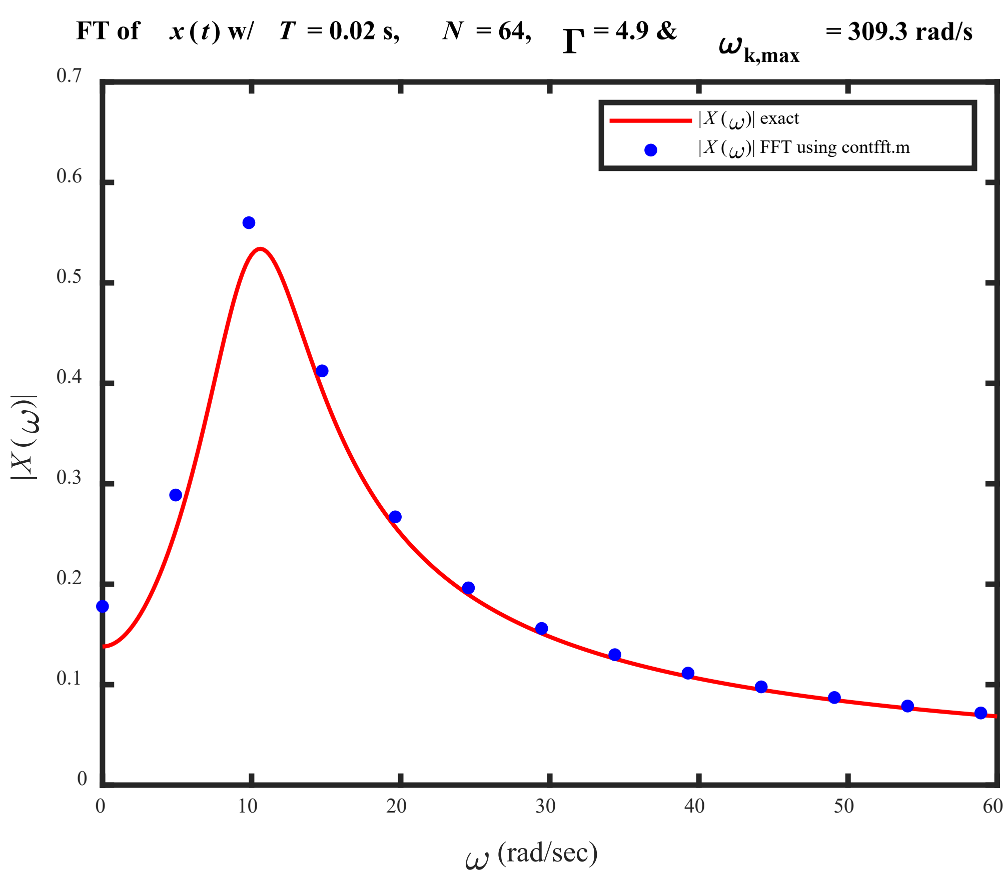
(i) $T = 0.04$ s and $N = 32$ ⇒ $\Gamma = \Delta\omega_k = 4.9087$ rad/s and $\omega_{k,\max} = 152.17$ rad/s



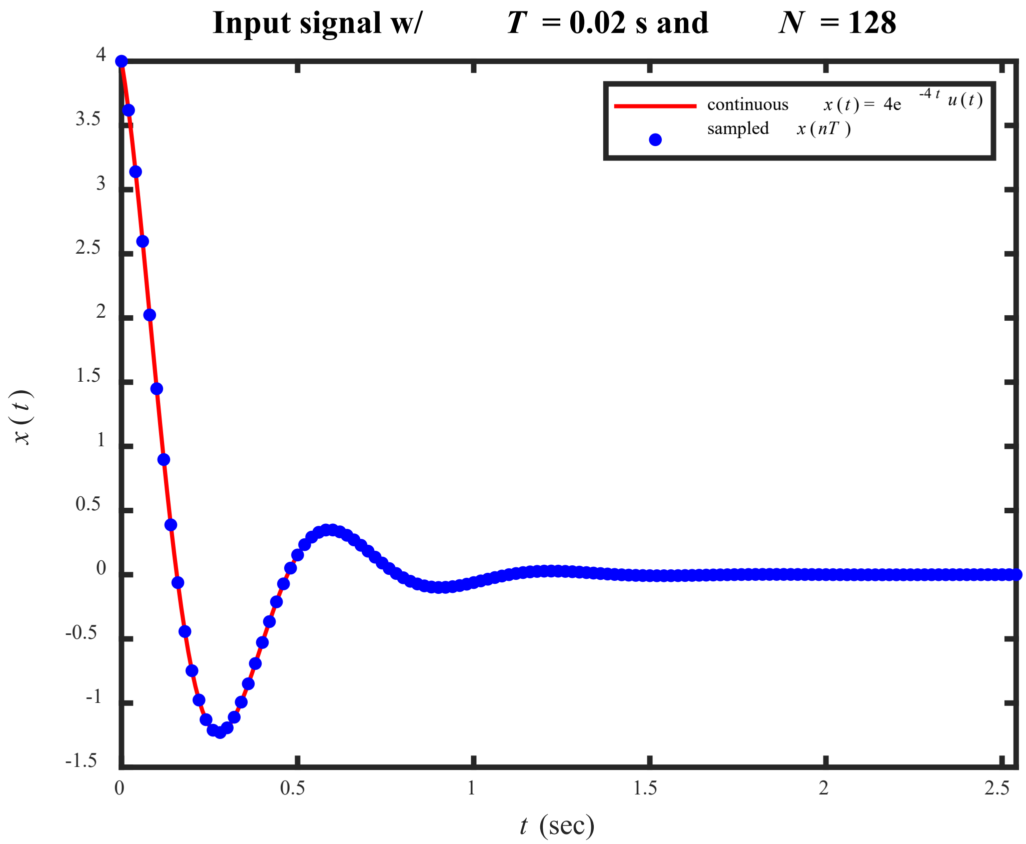


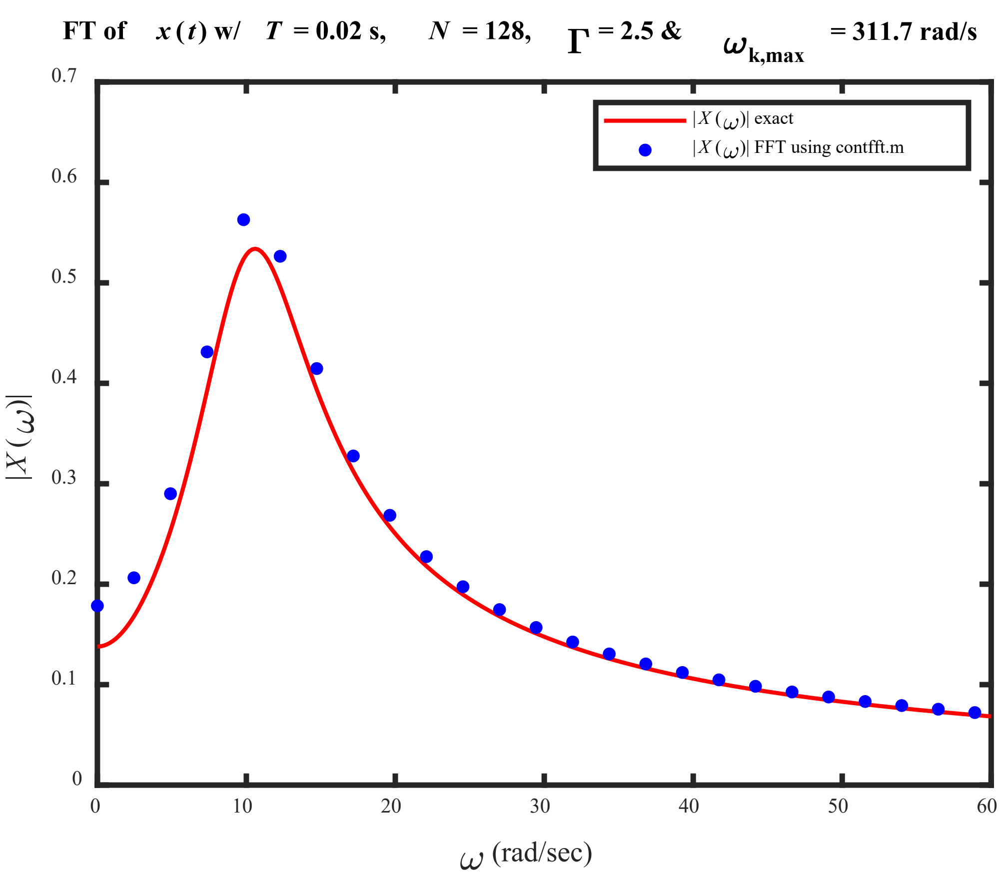
(ii) $T = 0.02$ s and $N = 64$ $\Rightarrow$ $\Gamma = \Delta\omega_k = 4.9087$ rad/s and $\omega_{k,\max} = 309.25$ rad/s



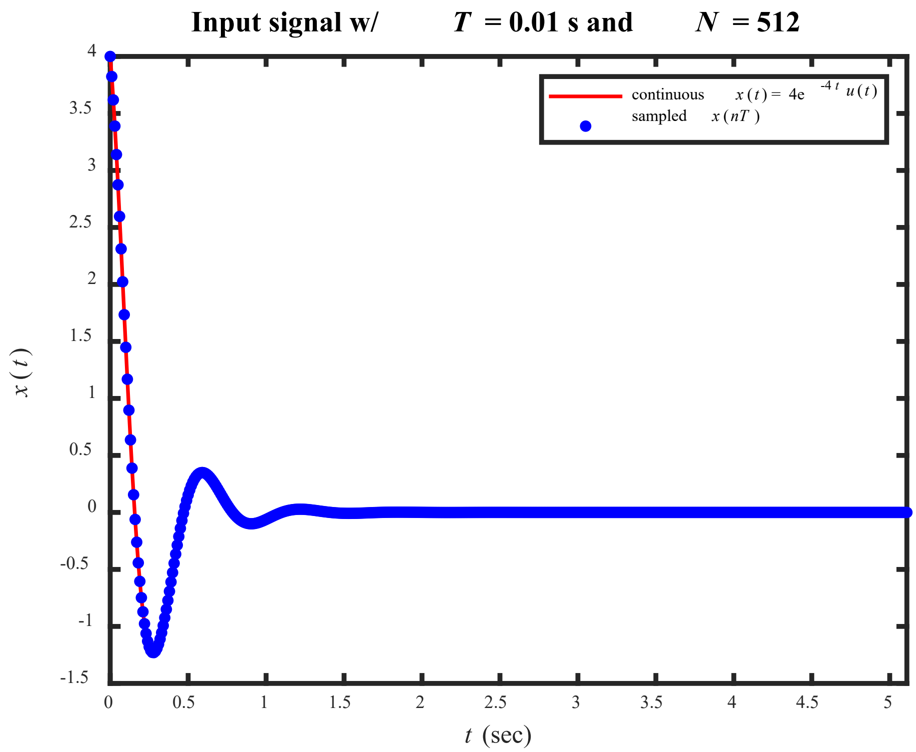


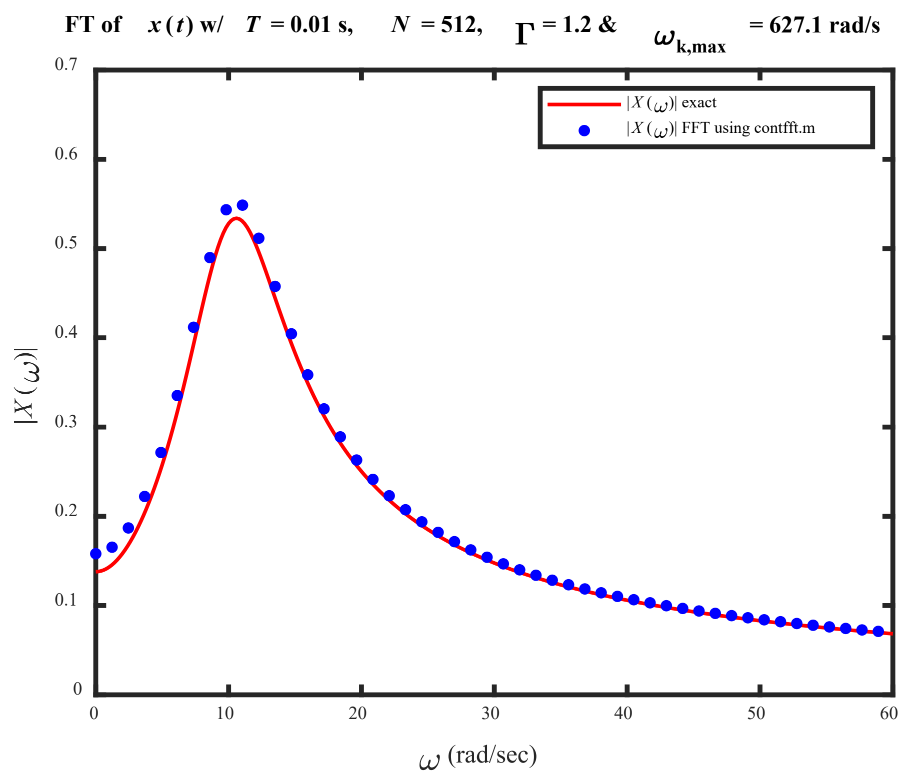
(iii) $T = 0.02$ s and $N = 128$ $\Rightarrow$ $\Gamma = \Delta\omega_k = 2.4544$ rad/s and $\omega_{k,max} = 311.705$ rad/s





(iv) $T = 0.01$ s and $N = 512$ ⇒ $\Gamma = \Delta\omega_k = 1.2272$ rad/s and $\omega_{k,max} = 627.09$ rad/s





c)

- Biggest gains in accuracy occur as T gets smaller; it is important to sample tightly for $0 \le t < 1$ s where $x(t)$ is changing rapidly.
- For $t > 1$ s, where $x(t) \approx 0$, increasing N only yields minimal improvements.
- Increasing N does decrease $\Gamma = \Delta\omega_k$, yielding 'nicer' looking plots of $|X(\omega_k)|$.