EE 313 Signals and Systems (Fall 2024) Examination 3

Name Key A

Instructions: **Show all work for full credit**. Write answers in indicated places. Insert equation sheet inside exam.

1) A voltage $v_i(t) = 16 + 7\cos(\omega_0 t) + 6.4\cos(2\omega_0 t)$ (V) $-\infty < t < \infty$ is input into a circuit with a frequency response $H(\omega) = \dfrac{V_o(\omega)}{V_i(\omega)} = \dfrac{R_L}{R_s + R_L + 1/j\omega C}$. Find the output $v_o(t)$ voltage when $\omega_0 = 100\pi$ rad/s, $R_s = 50\ \Omega$, $R_L = 150\ \Omega$, and $C = 20\ \mu\text{F}$. As part of the solution process evaluate $H(\omega)$ at $\omega = 0$, ω_0, and $2\omega_0$ (polar form w/ angle in radians).

$$H(\omega) = \frac{j\omega C R_L}{(R_s + R_L) j\omega C + 1} = \frac{j\omega 150(20\times10^{-6})}{(50+150)j\omega 20\times10^{-6} + 1} = \frac{j\omega 0.003}{1 + j\omega 0.004}$$

$$H(0) = 0$$

$$H(100\pi) = \frac{j100\pi(0.003)}{1 + j100\pi(0.004)} = 0.58686\angle 0.67216$$

$$H(200\pi) = \frac{j200\pi(0.003)}{1 + j200\pi(0.004)} = 0.69686\angle 0.37868$$

Per (5.11) $y(t) = A|H(\omega_0)|\cos(\omega_0 t + \theta + \measuredangle H(\omega_0))$.

Therefore,

$$v_o(t) = 16(0) + 7(0.58686)\cos(100\pi t + 0.67216) + 6.4(0.69686)\cos(200\pi t + 0.37868)\ \text{V}$$

$$= 4.108\cos(100\pi t + 0.67216) + 4.46\cos(200\pi t + 0.3787)\ \text{V}$$

$H(0) =$ 0 $\quad H(\omega_0) =$ $0.587\angle 0.672$ $\quad H(2\omega_0) =$ $0.697\angle 0.379$

$v_o(t) =$ $4.108\cos(100\pi t + 0.6722) + 4.46\cos(200\pi t + 0.3787)$ V $\quad -\infty < t < \infty$

2) The transfer function of a linear, time-invariant, casual (no initial energy), discrete-time filter is given by

$$H(z) = \frac{z^2 + 0.2z + 0.27}{z^2 + 0.36}.$$

First, find the partial fraction expansion of $H(z)/z$. Then, find an entirely real, analytic expression for the unit-pulse response $h[n]$ of the system for all $n \geq 0$. Calculate $h[0]$, $h[1]$, and $h[3]$.

$z^2 + 0.36 = 0 \Rightarrow z = \sqrt{-0.36} \Rightarrow z = \pm j0.6$

$$\frac{H(z)}{z} = \frac{z^2+0.2z+0.27}{z(z-j0.6)(z+j0.6)} = \frac{c_0}{z} + \frac{c_1}{z-j0.6} + \frac{c_2}{z+j0.6}$$

$p_1 = j0.6 = 0.6\angle\frac{\pi}{2}$

Residues $c_0 = H(0) = \frac{0.27}{0.36} = 0.75$

$$c_1 = \left\{(z-j0.6)\frac{z^2+0.2z+0.27}{z(z-j0.6)(z+j0.6)}\right\}\Bigg|_{z=j0.6} = \frac{(j0.6)^2+0.2(j0.6)+0.27}{j0.6(j0.6+j0.6)}$$

$$= 0.208\bar{3}\angle -0.9273 = 0.125 - j0.1\bar{6}\bar{6}$$

$$c_2 = c_1^* = 0.208\bar{3}\angle +0.9273 = 0.125 + j0.1\bar{6}\bar{6}$$

$$\frac{H(z)}{z} = \frac{0.75}{z} + \frac{0.208\bar{3}\angle -0.9273}{z-j0.6} + \frac{0.208\bar{3}\angle 0.9273}{z+j0.6}$$

$$H(z) = 0.75 + \frac{0.208\bar{3}\angle -0.9273\, z}{z-j0.6} + \frac{0.208\bar{3}\angle 0.9273\, z}{z+j0.6}$$

Use linearity, Table 7.3 $\delta[n] \leftrightarrow 1$, and (7.68)

$2|c_1|\sigma^n \cos(\Omega n + \measuredangle c_1)$ where $\sigma = |p_1| = 0.6$ & $\Omega = \measuredangle p_1 = +\frac{\pi}{2}$ $n \geq 0$

$$h[n] = 0.75\delta[n] + 2(0.208\bar{3})\,0.6^n \cos(+\tfrac{\pi}{2}n - 0.9273)\,u[n]$$

$\cos(-A) = \cos(A)$

$$h[n] = 0.75\delta[n] + 0.41\bar{6}\,(0.6)^n \cos(\tfrac{\pi}{2}n - 0.9273)\,u[n]$$

$h[0] = 0.75 + 0.41\bar{6}\cos(-0.9273) = 1$

$h[1] = 0.41\bar{6}(0.6)\cos(\frac{\pi}{2} - 0.9273) = 0.2$

$h[3] = 0.41\bar{6}(0.6)^3\cos(1.5\pi - 0.93) = -0.072$

$H(z)/z = \dfrac{0.75}{z} + \dfrac{0.208\bar{3}\angle -0.9273}{z-j0.6} + \dfrac{0.208\bar{3}\angle -0.9273}{z+j0.6}$

$h[n] = 0.75\delta[n] + 0.41\bar{6}\,(0.6)^n \cos(\frac{\pi}{2}n - 0.9273)\,u[n]$

$h[0] =$ 1 $h[1] =$ 0.2 $h[3] =$ −0.072

3) Given signal $x(t) = 8 + 4\sin(300t), -\infty < t < \infty$, find the frequency spectrum $X(\omega)$ and draw a fully labeled sketch of the magnitude of $|X(\omega)|$ on provided axes. If $x(t)$ is to be sampled, determine the maximum sampling rate T_{max} and corresponding minimum sampling frequency ω_{min} to avoid aliasing. If $x(t)$ is then sampled at a rate $T = \pi/250$ s = 12.5664 ms, does the sampled signal $x[n] = x(nT)$ experience aliasing? For $x[n]$, draw a fully labeled sketch of the magnitude of the frequency spectrum $|X_S(\omega)|$ on the provided axis for the frequency range shown.

Using linearity + Table 3.2 $\quad 1 \leftrightarrow 2\pi\delta(\omega)$

$\sin(\omega_0 t) \leftrightarrow j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$

$$X(\omega) = 8(2\pi\delta(\omega)) + 4j\pi[\delta(\omega+300) - \delta(\omega-300)]$$

From $|X(\omega)|$, $B = 300 \frac{rad}{s} \Rightarrow \omega_{min} = 2B = \underline{600 \frac{rad}{s}}$

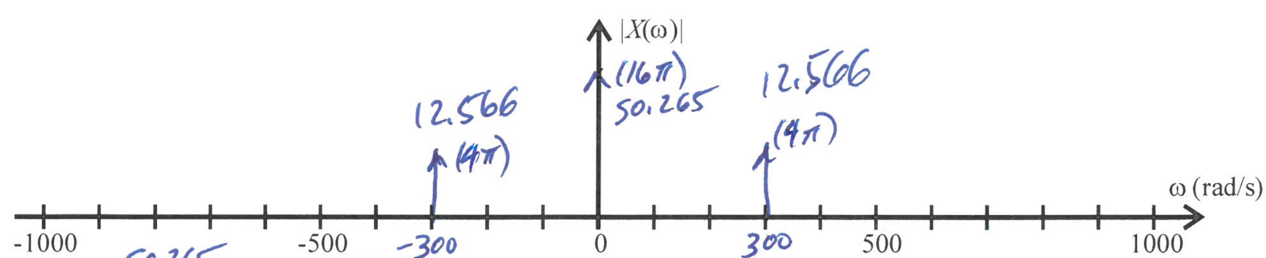
$$T_{max} = \frac{2\pi}{\omega_{min}} = \frac{2\pi}{600} = \pi/300 = 0.010472 \text{ s} = \underline{10.472 \text{ ms}}$$

Since $T = 12.5664$ ms $> T_{max} \Rightarrow$ aliasing!

Per (5.51), $X_s(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega - k\omega_s)$ $\qquad T = \pi/250 \quad \omega_s = \frac{2\pi}{T} = 500 \frac{rad}{s}$

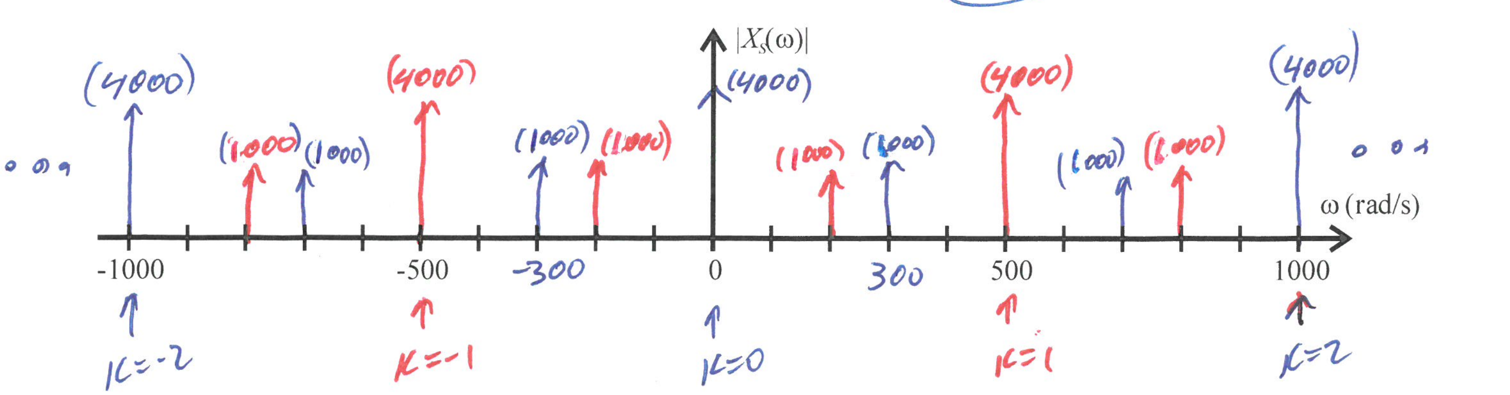
$\frac{1}{T} = \frac{250}{\pi}$

$16\pi\left(\frac{250}{\pi}\right) = 4000 \quad + \quad 4\pi\left(\frac{250}{\pi}\right) = 1000$



$X(\omega) = \underline{\overset{50.265}{16\pi\delta(\omega)} + j\overset{12.566}{4\pi}[\delta(\omega+300) - \delta(\omega-300)]}$ $\qquad \omega_{min} = \underline{600 \text{ rad/s}}$

$T_{max} = \underline{10.472 \text{ ms}}$ $\qquad$ If $T = 12.5664$ ms, is there aliasing? YES / NO (circle correct answer)



4) The transfer function of a linear, time-invariant, casual (no initial energy), discrete-time filter is given by

$$H(z) = \frac{z+0.5}{z^2 - 0.1z + 0.72}.$$

a) For an input $x[n] = 6u[n]$, find $X(z)$ and the output $Y(z)$ of the system as a rational function.

Use linearity and Table 7.3 $u[n] \Leftrightarrow \frac{z}{z-1}$, to get $X(z) = \frac{6z}{z-1}$. Per Table 7.2 convolution property

$$Y(z) = X(z)H(z) = \frac{6z(z+0.5)}{(z-1)(z^2-0.1z+0.72)} = \frac{6z^2+3z}{(z^3-0.1z^2+0.72z)-(z^2-0.1z+0.72)}$$

$=$

$X(z) =$ $\frac{6z}{z-1}$ $\qquad$ $Y(z) =$ $\frac{6z^2+3z}{z^3-1.1z^2+0.82z-0.72}$

b) Without solving for $y[n]$, determine $y[0]$ and $y[1]$

Per Initial-Value Theorem (Table 7.2)

$$x[0] = \lim_{z\to\infty} X(z) \Rightarrow y[0] = \lim_{z\to\infty} \frac{6/z + 3/z^2}{1 - \ldots} = \underline{\underline{0}}$$

$$x[1] = \lim_{z\to\infty} [zX(z) - z\,x[0]] \Rightarrow y[1] = \lim_{z\to\infty} \frac{6 + 3/z}{1-\ldots} = \underline{\underline{6}}$$

$y[0] =$ 0 $\qquad$ $y[1] =$ 6

c) Without solving for $y[n]$, determine $y[n \to \infty]$.

$z^2 - 0.1z + 0.72 \Rightarrow$ poles are $z = 0.8485\angle \pm 1.5118$ $\quad |z| < 1$

Final-value Theorem Table 7.3

$$\lim_{n\to\infty} y[n] = (z-1)Y(z)\Big|_{z=1} = \frac{\cancel{(z-1)}(6z^2+3z)}{\cancel{(z-1)}(z^2-0.1z+0.72)}$$

$$= \frac{6(1)^2+3(1)}{1^2-0.1(1)+0.72} = \frac{9}{1.62}$$

$y[n \to \infty] =$ $5.\overline{55}$