EE 313 Signals and Systems (Fall 2024) Examination 2

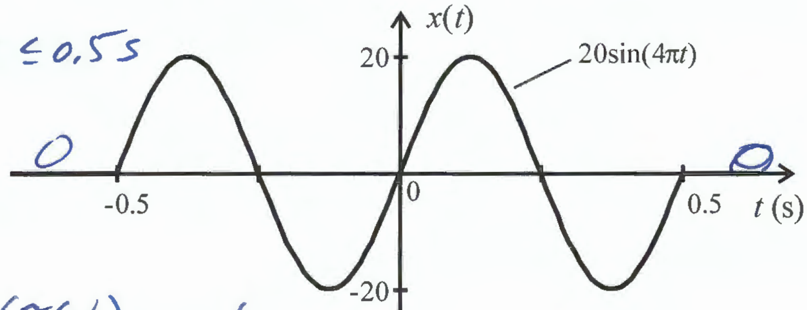
Name Key B

Instructions: Show all work for full credit. Write answers in indicated places. Insert equation sheet inside exam.

1) a) Write an analytic expression for $x(t)$. Then, compute the Fourier transform of $x(t)$.

$x(t) = 20\sin(4\pi t) \quad -0.5 \le t \le 0.5\,\text{s}$

$= 20\sin(4\pi t)\,p_1(t)$



use: linearity

$p_\tau(t) \leftrightarrow \tau\,\text{sinc}\left(\frac{\tau\omega}{2\pi}\right) \quad w/\ \tau = 1\text{s}$

$x(t)\sin(\omega_0 t) \leftrightarrow j/2\left[X(\omega+\omega_0) - X(\omega-\omega_0)\right] \quad w/\ \omega_0 = 4\pi\ \text{rad/s}$

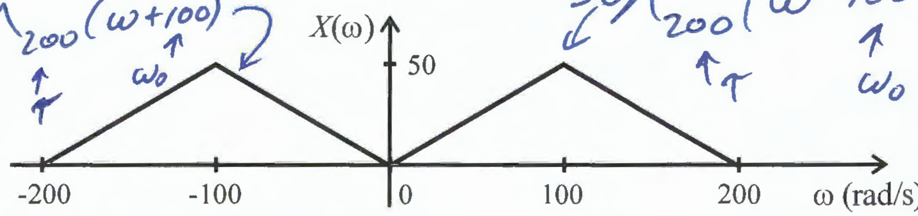
$X(\omega) = 20\ j/2\left[1\,\text{sinc}\left(\frac{1(\omega+4\pi)}{2\pi}\right) - 1\,\text{sinc}\left(\frac{1(\omega-4\pi)}{2\pi}\right)\right]$

$x(t) =$ $20\sin(4\pi t)\,p_1(t)$

$X(\omega) =$ $j10\left[\text{sinc}\left(\frac{\omega+4\pi}{2\pi}\right) - \text{sinc}\left(\frac{\omega-4\pi}{2\pi}\right)\right] \quad -\infty < \omega < \infty$

b) Compute the inverse Fourier transform of the signal shown.

$50\Lambda_{200}(\omega+100)$ (τ, ω_0) $\qquad 50\Lambda_{200}(\omega-100)$ (τ, ω_0)



use: linearity

$\frac{\tau}{2}\text{sinc}^2\left(\frac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\Lambda_\tau(\omega) \quad w/\ \tau = 200\ \text{rad/s}$

$x(t)\cos(\omega_0 t) \leftrightarrow 1/2\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right] \quad w/\ \omega_0 = 100\ \text{rad/s}$

$X(\omega) = 50\Lambda_{200}(\omega+100) + 50\Lambda_{200}(\omega-100)$

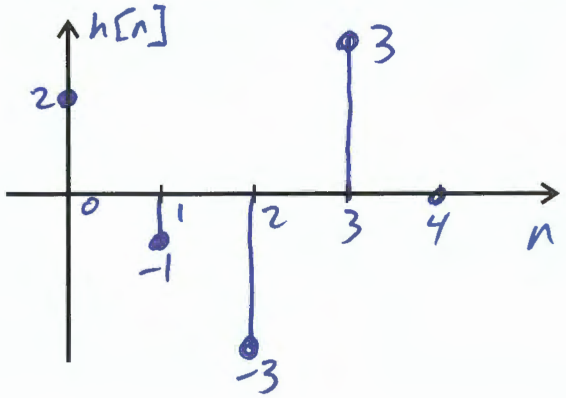
$= \frac{50(2)}{2\pi}\ 1/2\left[2\pi\Lambda_{200}(\omega+100) + 2\pi\Lambda_{200}(\omega-100)\right]$

$= \frac{50}{\pi}\ \frac{200}{2}\ \text{sinc}^2\left(\frac{200t}{4\pi}\right)\cos(100t)$

$x(t) =$ $\frac{5000}{\pi}\text{sinc}^2\left(\frac{50t}{\pi}\right)\cos(100t) \quad -\infty < t < \infty$

$= 1591.55\,\text{sinc}^2(15.9155t)\cos(100t) \quad -\infty < t < \infty$

2) A discrete-time unit pulse response of a system is $h[n] = 2\delta[n] - 1\delta[n-1] - 3\delta[n-2] + 3\delta[n-3]$. Sketch a stem plot of $h[n]$. How many non-zero points N are in $h[n]$? Then, compute the discrete Fourier transform of $h[n]$, write answers in box in **phasor form** w/ angle in degrees, i.e., $|H|\angle H$.



$0 \le n \le 3 \Rightarrow N = 4$

$$H_K = \sum_{n=0}^{N-1} h[n] e^{-j\frac{2\pi K n}{N}} \quad 0 \le K \le N-1$$

$$= \sum_{n=0}^{3} h[n] e^{-j\frac{2\pi K n}{4}} \quad 0 \le K \le 3$$

$K=0 \quad H_0 = h[0] + h[1] + h[2] + h[3] = 2 - 1 - 3 + 3 = \underline{\underline{1}}$

$K=1 \quad H_1 = 2 - 1e^{-j\frac{\pi}{2}(1)(1)} - 3e^{-j\frac{\pi}{2}(1)(2)} + 3e^{-j\frac{\pi}{2}(1)(3)} = \underline{\underline{6.403\angle 38.66^\circ}}$

$K=2 \quad H_2 = 2 - 1e^{-j\frac{\pi}{2}(2)(1)} - 3e^{-j\frac{\pi}{2}(2)(2)} + 3e^{-j\frac{\pi}{2}(2)(3)} = \underline{\underline{-3}}$

$K=3 \quad H_3 = 2 - 1e^{-j\frac{\pi}{2}(3)1} - 3e^{-j\frac{\pi}{2}(3)2} + 3e^{-j\frac{\pi}{2}3(3)} = \underline{\underline{6.403\angle -38.66^\circ}}$

$N =$ 4

$$\boxed{H_0 = 1\angle 0^\circ,\ H_1 = 6.403\angle 38.66^\circ,\ H_2 = 3\angle \pm 180^\circ,\ H_3 = 6.403\angle -38.66^\circ}$$

3) Given the discrete-time signal $x[n] = \text{sgn}[n]$, find the discrete-time Fourier transform of:

a) $v[n] = 8x[n+4]$ Per Table 4.1, $\text{sgn}[n] \leftrightarrow \frac{2}{1-e^{-j\Omega}}$

Table 4.2 linearity $ax[n] \leftrightarrow aX(\Omega)$

R/L shift $x[n-q] \leftrightarrow X(\Omega)e^{-jq\Omega}$ w/ $q=-4$

So,

$$V(\Omega) = 8\,\frac{2}{1-e^{-j\Omega}}\,e^{+j4\Omega}$$

$$\underline{V(\Omega) = \frac{16}{1-e^{-j\Omega}}\,e^{j4\Omega} \quad -\infty<\Omega<\infty}$$

b) $w[n] = 3x[n]\sin[4n]$

Table 4.2 linearity $ax(n) \leftrightarrow aX(\Omega)$

mult by $\sin \Omega_0 n$ $x[n]\sin\Omega_0 n \leftrightarrow \frac{j}{2}\left[X(\Omega+\Omega_0) - X(\Omega-\Omega_0)\right]$

w/ $\Omega_0 = 4$

$$W(\Omega) = 3\,\frac{j}{2}\left[\frac{2}{1-e^{-j(\Omega+4)}} - \frac{2}{1-e^{-j(\Omega-4)}}\right]$$

$$\underline{W(\Omega) = j3\left[\frac{1}{1-e^{-j(\Omega+4)}} - \frac{1}{1-e^{-j(\Omega-4)}}\right] \quad -\infty<\Omega<\infty}$$

c) $y[n] = x[-n] * x[n]$

Table 4.2 conv. in time-domain $x[n]*v[n] \leftrightarrow X(\Omega)V(\Omega)$

Time-reversal $x[-n] \leftrightarrow X(-\Omega) = X^*(\Omega)$

$$X[-n] \leftrightarrow X(-\Omega) = \frac{2}{1-e^{+j\Omega}}$$

$$Y(\Omega) = X(-\Omega)X(\Omega) = \frac{2}{1-e^{+j\Omega}}\,\frac{2}{1-e^{j\Omega}}$$

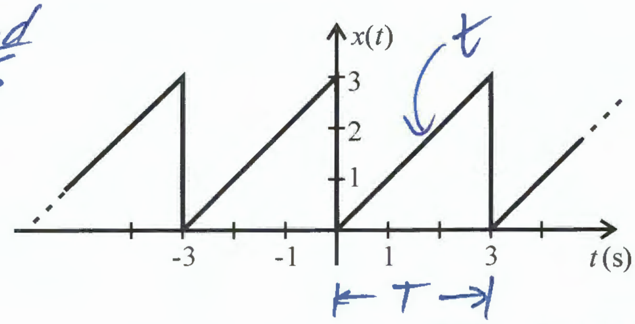
Note: $(1-e^{j\Omega})(1-e^{-j\Omega}) = 1 - e^{-j\Omega} - e^{j\Omega} + \underbrace{e^0}_{1} = 2-2\cos\Omega$

$$\underline{Y(\Omega) = \frac{4}{(1-e^{j\Omega})(1-e^{-j\Omega})} \quad -\infty<\Omega<\infty}$$

$$= \frac{2}{1-\cos\Omega} \quad -\infty<\Omega<\infty$$

4) For the signal shown below, we wish to consider the trigonometric Fourier series. Is $x(t)$ even, odd, or neither? What are the fundamental period (s) and frequency (rad/s) of $x(t)$? Then, calculate the coefficients a_0, a_1, and b_1. Also, find amplitude coefficients A_0 and A_1. Last, find the total power in $x(t)$ as well as the power in the DC and first harmonic components.

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3} = 2.0944\ \frac{rad}{s}$$



$x(t)$ even, odd, or neither? (circle correct answer)

fundamental period = 3 s   fundamental frequency = $2\pi/3 = 2.0944\ \frac{rad}{s}$

$$(3.7)\quad a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{3}\int_0^3 t\,dt = \frac{1}{3}\left.\frac{t^2}{2}\right|_0^3$$

$$= \frac{1}{6}(9-0) = \underline{1.5}$$

$$(3.5)\quad a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$a_1 = \frac{2}{3}\int_0^3 t\cos\left(\frac{2\pi}{3}t\right)dt = \frac{2}{3}\frac{1}{(\frac{2\pi}{3})^2}\left[\cos\left(\frac{2\pi}{3}t\right) + \frac{2\pi}{3}t\sin\left(\frac{2\pi}{3}t\right)\right]\Bigg|_0^3$$

$$= \frac{2}{3}\frac{1}{(2\pi/3)^2}\left\{\left(\cos(2\pi) + 2\pi\sin(2\pi)\right) - \left(\cos(0) + \frac{2\pi}{3}0\sin(0)\right)\right\}$$

$$= \frac{2}{3}\frac{1}{(2\pi/3)^2}\left[(1+0)-(1+0)\right] = \underline{0}$$

$$(3.6)\quad b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$b_1 = \frac{2}{3}\int_0^3 t\sin\left(\frac{2\pi}{3}t\right)dt = \frac{2}{3}\frac{1}{(2\pi/3)^2}\left[\sin\left(\frac{2\pi}{3}t\right) - \frac{2\pi}{3}t\cos\left(\frac{2\pi}{3}t\right)\right]\Bigg|_0^3$$

$$= \frac{2}{3}\frac{1}{(2\pi/3)^2}\left\{\left(\sin(2\pi) - 2\pi\cos(2\pi)\right) - \left(\sin(0) - \frac{2\pi}{3}0\cos(0)\right)\right\}$$

$$= \frac{\cancel{2}}{\cancel{3}}\frac{1}{\frac{\cancel{2\pi}}{\cancel{3}}\frac{2\pi}{3}}\left[(0-\cancel{2}\pi)-(0-0)\right] = \underline{-\frac{3}{\pi} = -0.95493}$$

4) cont.

$$(3.9)\quad A_k = \sqrt{a_k^2 + b_k^2}$$

$$A_0 = a_0 = \underline{1.5}$$

$$A_1 = \sqrt{0^2 + (-3/\pi)^2} = \underline{3/\pi = 0.95493}$$

$$(3.28)\quad P = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{3}\int_0^3 t^2\,dt = \frac{1}{3}\left.\frac{t^3}{3}\right|_0^3$$

$$= 1/9\,(27-0) = \underline{3}$$

From Notes: $P = a_0^2 + 1/2 \sum_{k=1}^{\infty} A_k^2$

$$P_{DC} = a_0^2 = 1.5^2 = \underline{2.25}$$

$$P_1 = 1/2\, A_1^2 = 1/2\,(3/\pi)^2 = \frac{9}{2\pi^2} = \underline{0.455945}$$

$a_0 =$ 1.5 $\quad a_1 =$ 0 $\quad b_1 =$ $\frac{-3}{\pi} = -0.95493$

$A_0 =$ 1.5 $\quad A_1 =$ $\frac{3}{\pi} = 0.95493$

$P_{x(t)} =$ 3 $\quad P_{DC} =$ 2.25 $\quad P_{first} =$ 0.455945

Useful Integrals

$\int \sin ax\,dx = \frac{-1}{a}\cos ax$, $\int \cos ax\,dx = \frac{1}{a}\sin ax$, $\int \sin ax \cos ax\,dx = \frac{1}{2a}\sin^2 ax$, $\int \sin^2 ax\,dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$,

$\int \frac{1}{x}dx = \ln|x|$, $\int \cos^2 ax\,dx = \frac{1}{2}x + \frac{1}{4a}\sin 2ax$, $\int x \sin ax\,dx = \frac{1}{a^2}[\sin ax - ax\cos ax]$, $\int e^{ax}dx = \frac{e^{ax}}{a}$,

$\int x\cos ax\,dx = \frac{1}{a^2}[\cos ax + ax\sin ax]$, $\int xe^{ax}dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$, $\int x^2 e^{ax}dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right)$