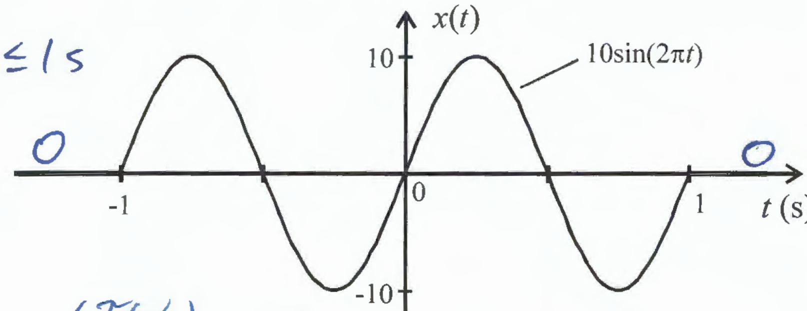
EE 313 Signals and Systems (Fall 2024) Examination 2

Name Key A

Instructions: Show all work for full credit. Write answers in indicated places. Insert equation sheet inside exam.

1) a) Write an analytic expression for $x(t)$. Then, compute the Fourier transform of $x(t)$.



$x(t) = 10\sin(2\pi t) \quad -1 \le t \le 1\,s$

$= 10\sin(2\pi t)\, p_2(t)$

Use: linearity

$p_\tau(t) \leftrightarrow \tau\, \text{sinc}\left(\frac{\tau\omega}{2\pi}\right)$ w/ $\tau = 2s$

$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(\omega+\omega_0) - X(\omega-\omega_0)\right]$ w/ $\omega_0 = 2\pi \frac{rad}{s}$

$X(\omega) = 10\,\frac{j}{2}\left[2\,\text{sinc}\left(\frac{2(\omega+2\pi)}{2\pi}\right) - 2\,\text{sinc}\left(\frac{2(\omega-2\pi)}{2\pi}\right)\right]$

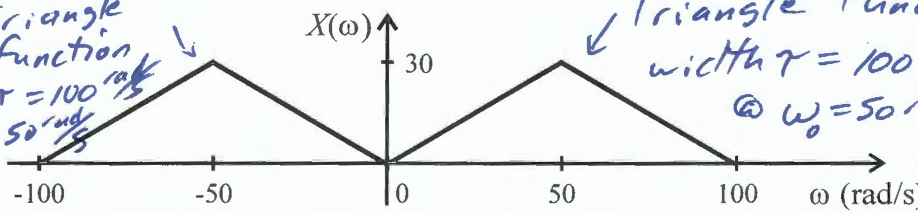
$x(t) =$ $10\sin(2\pi t)\, p_2(t)$

$X(\omega) =$ $j10\left[\text{sinc}\left(\frac{\omega+2\pi}{\pi}\right) - \text{sinc}\left(\frac{\omega-2\pi}{\pi}\right)\right] \quad -\infty < \omega < \infty$

b) Compute the inverse Fourier transform of the signal shown.

Triangle function $\tau = 100 \frac{rad}{s}$ $\omega_0 = -50 \frac{rad}{s}$

Triangle function width $\tau = 100 \frac{rad}{s}$ @ $\omega_0 = 50 \frac{rad}{s}$



Use: linearity

$\frac{\tau}{2}\,\text{sinc}^2\left(\frac{\tau t}{4\pi}\right) \leftrightarrow 2\pi \Lambda_\tau(\omega)$ w/ $\tau = 100 \frac{rad}{s}$

$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$ w/ $\omega_0 = 50 \frac{rad}{s}$

$X(\omega) = 30\Lambda_{100}(\omega+50) + 30\Lambda_{100}(\omega-50)$

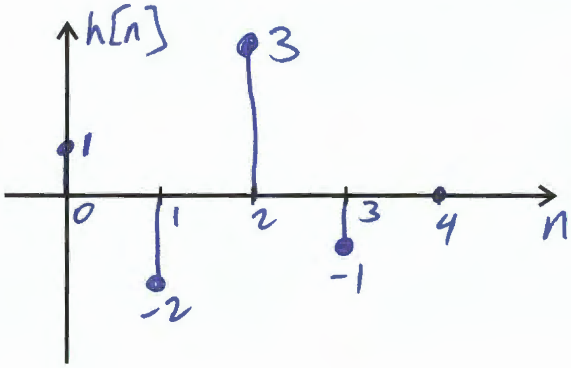
$= \frac{30(2)}{2\pi}\,\frac{1}{2}\left[2\pi\Lambda_{100}(\omega+50) + 2\pi\Lambda_{100}(\omega-50)\right]$

$x(t) = \frac{30}{\pi}\,\frac{100}{2}\,\text{sinc}^2\left(\frac{100t}{4\pi}\right)\cos(50t)$

$x(t) =$ $1500/\pi\ \text{sinc}^2\left(\frac{25t}{\pi}\right)\cos(50t) \quad -\infty < t < \infty$

$= 477.465\,\text{sinc}^2(7.96t)\cos(50t) \quad -\infty < t < \infty$

2) A discrete-time unit pulse response of a system is $h[n] = 1\delta[n] - 2\delta[n-1] + 3\delta[n-2] - 1\delta[n-3]$. Sketch a stem plot of $h[n]$. How many non-zero points N are in $h[n]$? Then, compute the discrete Fourier transform of $h[n]$, write answers in box in **phasor form** w/ angle in degrees, i.e., $|H|\angle H$.



$0 \le n \le 3 \Rightarrow N = 4$

$$H_K = \sum_{n=0}^{N-1} h[n] e^{-j\frac{2\pi K n}{N}} \qquad 0 \le K \le N-1$$

$$= \sum_{n=0}^{3} h[n] e^{-j\frac{2\pi K n}{4}} \qquad 0 \le K \le 3$$

$K=0 \quad H_0 = h[0] + h[1] + h[2] + h[3] = 1 - 2 + 3 - 1 = \underline{1}$

$K=1 \quad H_1 = 1 - 2e^{-j\frac{\pi}{2}(1)1} + 3e^{-j\frac{\pi}{2}1(2)} - 1e^{-j\frac{\pi}{2}1(3)} = \underline{2.2361\angle 153.435^\circ}$

$K=2 \quad H_2 = 1 - 2e^{-j\frac{\pi}{2}2(1)} + 3e^{-j\frac{\pi}{2}2(2)} - 1e^{-j\frac{\pi}{2}2(3)} = \underline{7}$

$K=3 \quad H_3 = 1 - 2e^{-j\frac{\pi}{2}3(1)} + 3e^{-j\frac{\pi}{2}3(2)} - 1e^{-j\frac{\pi}{2}3(3)} = \underline{2.2361\angle -153.435^\circ}$

$N = $ 4

$\boxed{H_0 = 1\angle 0^\circ,\ H_1 = 2.2361\angle 153.4^\circ,\ H_2 = 7\angle 0^\circ,\ H_3 = 2.2361\angle -153.4^\circ}$

3) Given the discrete-time signal $x[n] = \text{sgn}[n]$, find the discrete-time Fourier transform of:

a) $v[n] = 6x[n+3]$

Per Table 4.1, $\text{sgn}[n] \leftrightarrow \dfrac{2}{1-e^{-j\Omega}}$

Table 4.2

Use linearity $ax[n] \leftrightarrow aX(\Omega)$

Right/left shift $x[n-q] \leftrightarrow X(\Omega)e^{-jq\Omega}$ w/ $q=-3$

So, $V(\Omega) = 6\dfrac{2}{1-e^{-j\Omega}}e^{+j3\Omega}$

$$\underline{V(\Omega) = \frac{12}{1-e^{-j\Omega}}e^{j3\Omega} \quad -\infty<\Omega<\infty}$$

b) $w[n] = 4x[n]\sin[3n]$

Table 4.2

Use linearity $ax[n] \leftrightarrow aX(\Omega)$

Mult by $\sin\Omega_0 n$: $x[n]\sin\Omega_0 n \leftrightarrow \frac{j}{2}\left[X(\Omega+\Omega_0) - X(\Omega-\Omega_0)\right]$

w/ $\Omega_0 = 3$

$$W(\Omega) = 4\frac{j}{2}\left[\frac{2}{1-e^{-j(\Omega+3)}} - \frac{2}{1-e^{-j(\Omega-3)}}\right]$$

$$\underline{W(\Omega) = j4\left[\frac{1}{1-e^{-j(\Omega+3)}} - \frac{1}{1-e^{-j(\Omega-3)}}\right]}$$

$-\infty<\Omega<\infty$

c) $y[n] = x[n] * x[-n]$

Table 4.2 conv. in time-domain $x[n]*v[n] \leftrightarrow X(\Omega)V(\Omega)$

Time-reversal $x[-n] \leftrightarrow X(-\Omega) = X^*(\Omega)$

$$x[-n] \leftrightarrow \frac{2}{1-e^{+j\Omega}} = X(-\Omega)$$

$$Y(\Omega) = X(\Omega)X(-\Omega) = \frac{2}{1-e^{-j\Omega}}\frac{2}{1-e^{+j\Omega}}$$

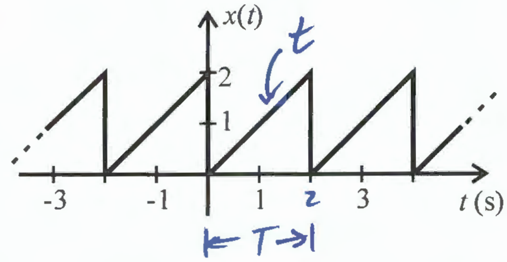
Note: $(1-e^{-j\Omega})(1-e^{j\Omega}) = 1 - e^{j\Omega} - e^{-j\Omega} + 1 = 2 - 2\cos\Omega$

$$\underline{Y(\Omega) = \frac{4}{(1-e^{-j\Omega})(1-e^{j\Omega})} \quad -\infty<\Omega<\infty}$$

$$= \frac{2}{1-\cos\Omega} \quad -\infty<\Omega<\infty$$

4) For the signal shown below, we wish to consider the trigonometric Fourier series. Is $x(t)$ even, odd, or neither? What are the fundamental period (s) and frequency (rad/s) of $x(t)$? Then, calculate the coefficients a_0, a_1, and b_1. Also, find amplitude coefficients A_0 and A_1. Last, find the total power in $x(t)$ as well as the power in the DC and first harmonic components.

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \ \frac{\text{rad}}{\text{s}}$$



$x(t)$ even, odd, or neither? (circle correct answer)

fundamental period = 2 s fundamental frequency = $\pi = 3.142 \ \text{rad/s}$

(3.7) $$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2}\int_0^2 t\,dt = \frac{1}{2}\,\frac{t^2}{2}\Big|_0^2$$

$$= \tfrac{1}{4}(4-0) = \underline{\underline{1}}$$

(3.5) $$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$a_1 = \frac{2}{2}\int_0^2 t\cos((1)\pi t)\,dt = \frac{1}{\pi^2}\left[\cos(\pi t) + \pi t\sin(\pi t)\right]\Big|_0^2$$

$$= \frac{1}{\pi^2}\left[\left(\cos(2\pi) + \pi 2\sin(2\pi)\right) - \left(\cos 0 + \pi\, 0 \sin 0\right)\right]$$

$$= \frac{1}{\pi^2}\left[(1+0) - (1+0)\right] = \underline{\underline{0}}$$

(3.6) $$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$$b_1 = \frac{2}{2}\int_0^2 t\sin(\pi t)\,dt = \frac{1}{\pi^2}\left[\sin(\pi t) - \pi t\cos(\pi t)\right]\Big|_0^2$$

$$= \frac{1}{\pi^2}\left[\left(\sin(2\pi) - 2\pi\cos(2\pi)\right) - \left(\sin(0) - \pi\, 0\cos(0)\right)\right]$$

$$= \frac{1}{\pi^2}\left[(0-2\pi) - (0-0)\right] = \underline{\underline{-2/\pi = -0.63662}}$$

4) cont.

(3.9) $A_k = \sqrt{a_k^2 + b_k^2}$

$A_0 = a_0 = \underline{1}$

$A_1 = \sqrt{0^2 + (-2/\pi)^2} = \underline{2/\pi = 0.63662}$

(3.28) $P = \frac{1}{T}\int_0^T x^2(t)\,dt = \frac{1}{2}\int_0^2 t^2\,dt = \frac{1}{2}\frac{t^3}{3}\Big|_0^2$

$= \frac{1}{6}(2^3 - 0) = 8/6 = \underline{\underline{1.\overline{3}}}$

From Notes: $P = a_0^2 + 1/2 \sum_{k=1}^{\infty} A_k^2$

$P_{DC} = a_0^2 = 1^2 = \underline{1}$

$P_1 = 1/2\, A_1^2 = 1/2\,(2/\pi)^2 = \frac{1}{2}\frac{4}{\pi^2} = \frac{2}{\pi^2} = \underline{0.20264}$

$a_0 =$ 1 $\qquad a_1 =$ 0 $\qquad b_1 = -\frac{2}{\pi} = -0.63662$

$A_0 =$ 1 $\qquad A_1 = 2/\pi = 0.63662$

$P_{x(t)} = 1.\overline{3}$ $\qquad P_{\mathrm{DC}} =$ 1 $\qquad P_{\mathrm{first}} = \frac{2}{\pi^2} = 0.20264$

Useful Integrals

$\int \sin ax\, dx = \frac{-1}{a}\cos ax\,, \quad \int \cos ax\, dx = \frac{1}{a}\sin ax\,, \quad \int \sin ax \cos ax\, dx = \frac{1}{2a}\sin^2 ax\,, \quad \int \sin^2 ax\, dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax\,,$

$\int \frac{1}{x}dx = \ln|x|\,, \quad \int \cos^2 ax\, dx = \frac{1}{2}x + \frac{1}{4a}\sin 2ax\,, \quad \int x \sin ax\, dx = \frac{1}{a^2}\Big[\sin ax - ax\cos ax\Big]\,, \quad \int e^{ax} dx = \frac{e^{ax}}{a}\,,$

$\int x \cos ax\, dx = \frac{1}{a^2}\Big[\cos ax + ax\sin ax\Big]\,, \quad \int x\, e^{ax} dx = \frac{e^{ax}}{a}\Big(x - \frac{1}{a}\Big)\,, \quad \int x^2 e^{ax} dx = \frac{e^{ax}}{a}\Big(x^2 - \frac{2x}{a} + \frac{2}{a^2}\Big)$