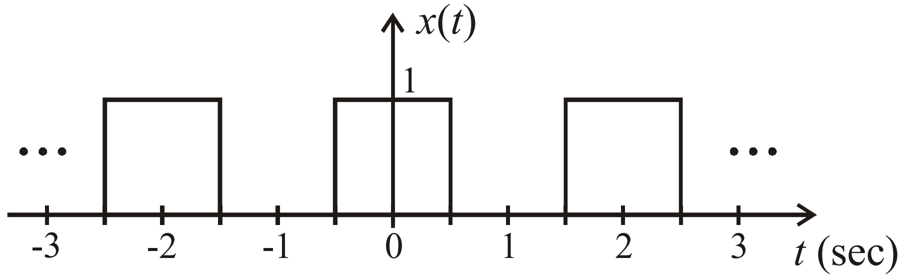
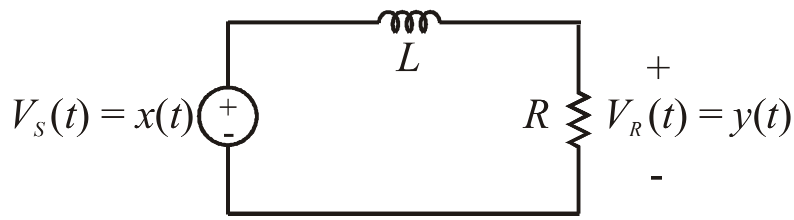
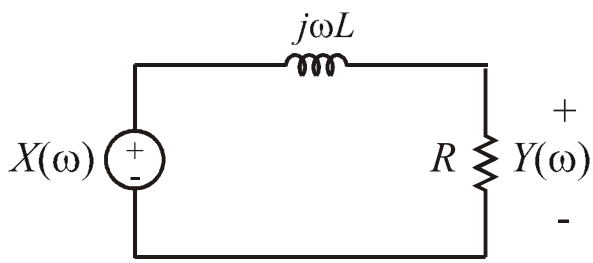
Consider the rectangular pulse train of example 3.2 of the text as the input to the series LR circuit.



Rectangular pulse train $x(t)$ input signal.



(a)



(b)

(a) Series LR circuit and (b) phasor equivalent circuit.

This pulse train $x(t)$, an even function, has a period of $T = 2$ s and fundamental frequency of $\omega_0 = \dfrac{2\pi}{T} = \pi$ (rad/s). The trigonometric Fourier series of $x(t)$ has coefficients

$$a_0 = 0.5 \text{ for } k = 0 \text{ (DC term)}$$

and

$$a_k = \frac{2}{k\pi}\sin(0.5k\pi) \;\; \& \;\; b_k = 0 \;\; \text{for} \;\; k = 1, 2, 3, \ldots \text{ (harmonics)}$$

yielding

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\pi t) = 0.5 + \sum_{k=1}^{\infty} \frac{2}{k\pi}\sin(0.5k\pi)\cos(k\pi t) \quad -\infty < t < \infty .$$

The cosine-with-phase trigonometric form of the trigonometric Fourier series for $x(t)$ has coefficients

$$a_0 = 0.5 \text{ for } k = 0 \text{ (DC term)},$$

$$A_k^x = |a_k| = \frac{2}{k\pi}|\sin(0.5k\pi)| \text{ for } k = 1,2,3,\ldots$$
$$= \begin{cases} 0 & k \text{ even} \\ \dfrac{2}{k\pi} & k \text{ odd} \end{cases},$$

and

$$\theta_k^x = \measuredangle a_k = \begin{cases} 0 & k = 0 \\ \measuredangle\left[\dfrac{2}{k\pi}\sin(0.5k\pi)\right] & k = 1,2,3,\ldots \end{cases} = \begin{cases} 0 & k \text{ even } \& \ k = 1,5,9,\ldots \\ \pi & k = 3,7,11,\ldots \ (a_k < 0) \end{cases}.$$

The cosine-with-phase trigonometric form of the trigonometric Fourier series for $x(t)$ is then

$$x(t) = a_0 + \sum_{k=1}^{\infty} A_k^x \cos(k\pi t + \theta_k^x)$$
$$= 0.5 + \sum_{k=1}^{\infty} \frac{2}{k\pi} |\sin(0.5k\pi)| \cos\left(k\pi t + \measuredangle(\frac{2}{k\pi}\sin(0.5k\pi)\right).$$

By voltage division, the frequency response $H(\omega)$ of the series LR circuit was found to be

$$H(\omega) = \frac{1}{1 + j\omega\left(L/R\right)}.$$

The cosine-with-phase trigonometric form of the trigonometric Fourier series for the system output $y(t)$ is then

$$y(t) = a_0 |H(0)| + \sum_{k=1}^{\infty} A_k^x |H(k\pi)| \cos\left(k\pi t + \theta_k^x + \measuredangle H(k\pi)\right)$$
$$= A_0^y + \sum_{k=1}^{\infty} A_k^y \cos\left(k\pi t + \theta_k^y\right)$$

where the cosine-with-phase Fourier series coefficients for the output $y(t)$ are

$$A_0^y = a_0 \left|H(0)\right| = a_0,$$

$$A_k^y = A_k^x \mid H(k\pi) \mid = A_k^x \left| \frac{1}{1 + j\omega\left(L/R\right)} \right|,$$

and

$$\theta_k^y = \theta_k^x + \measuredangle H(k\pi) = \theta_k^x + \measuredangle\left(\frac{1}{1 + j\omega\left(L/R\right)} \right).$$

For example, the table below shows the values of these various coefficients when the inductance to resistance ratio is ***L/R* = 1/π**.

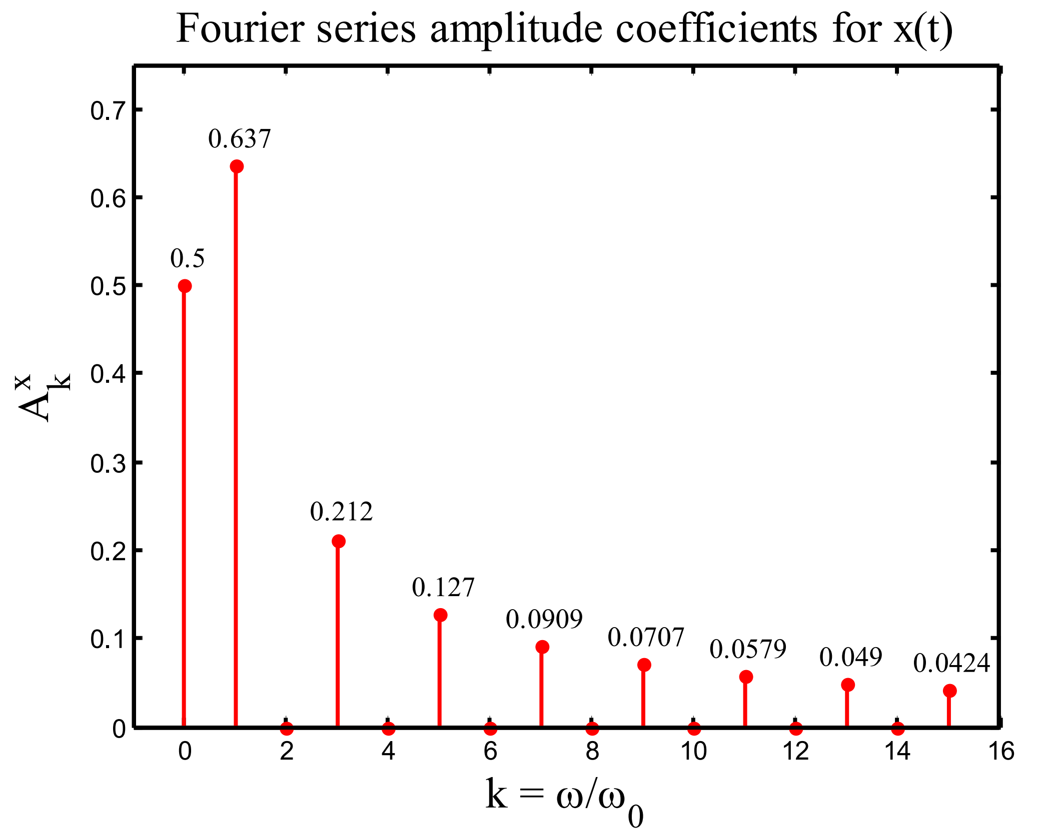
$k = \omega_k / \omega_0$ $\omega_0 = \pi$	$H(\omega_k) = \frac{1}{1 + jk\pi L/R}$	A_k^x	θ_k^x	A_k^y	θ_k^y
0	1	0.5	0	0.5	0 (n/a)
1	$\frac{1}{1+j1} = 0.707\angle -45°$	$\frac{2}{\pi}$	0	0.45	–45°
2	$\frac{1}{1+j2} = 0.447\angle -63.4°$	0	0	0	0 (n/a)
3	$\frac{1}{1+j3} = 0.316\angle -71.6°$	$\frac{2}{3\pi}$	180°	0.067	108.4°
4	$\frac{1}{1+j4} = 0.2425\angle -76°$	0	0	0	0 (n/a)
5	$\frac{1}{1+j5} = 0.196\angle -78.7°$	$\frac{2}{5\pi}$	0	0.025	–78.7°

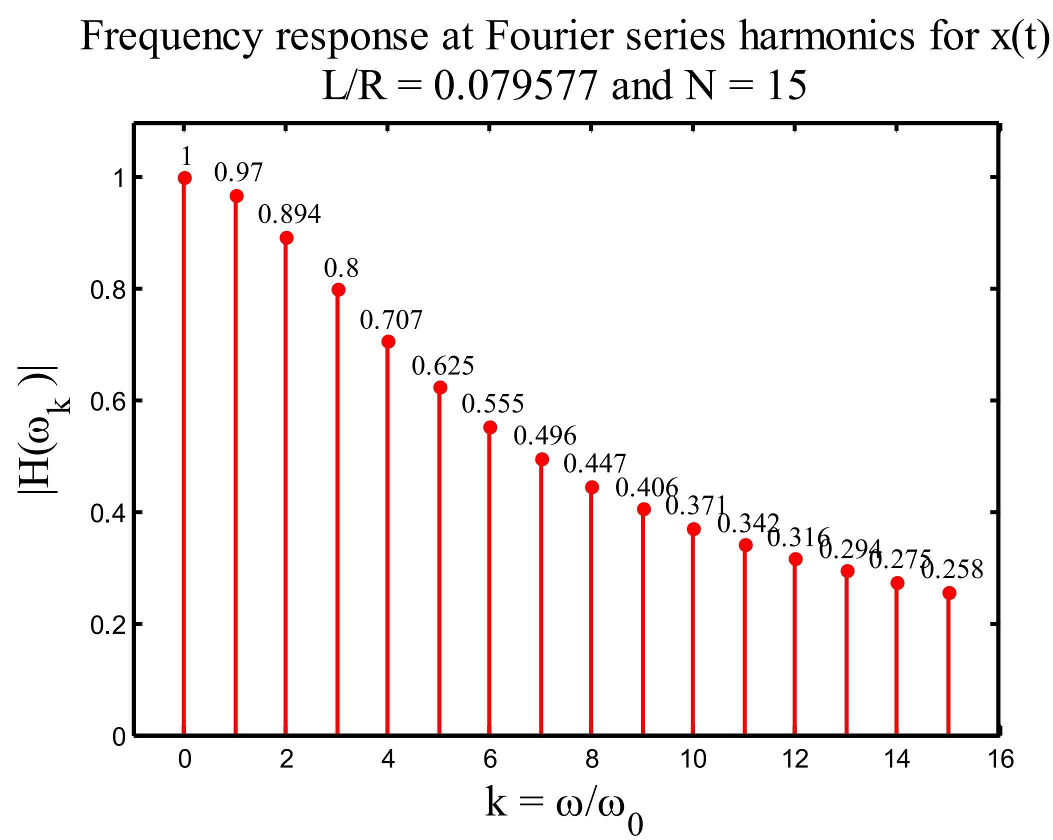
- Obviously, these calculations are much easier when a computer program (e.g., MATLAB) is used.
- Also, in the MATLAB file, we go to the N^{th} harmonic instead of ∞ in the Fourier series summation.

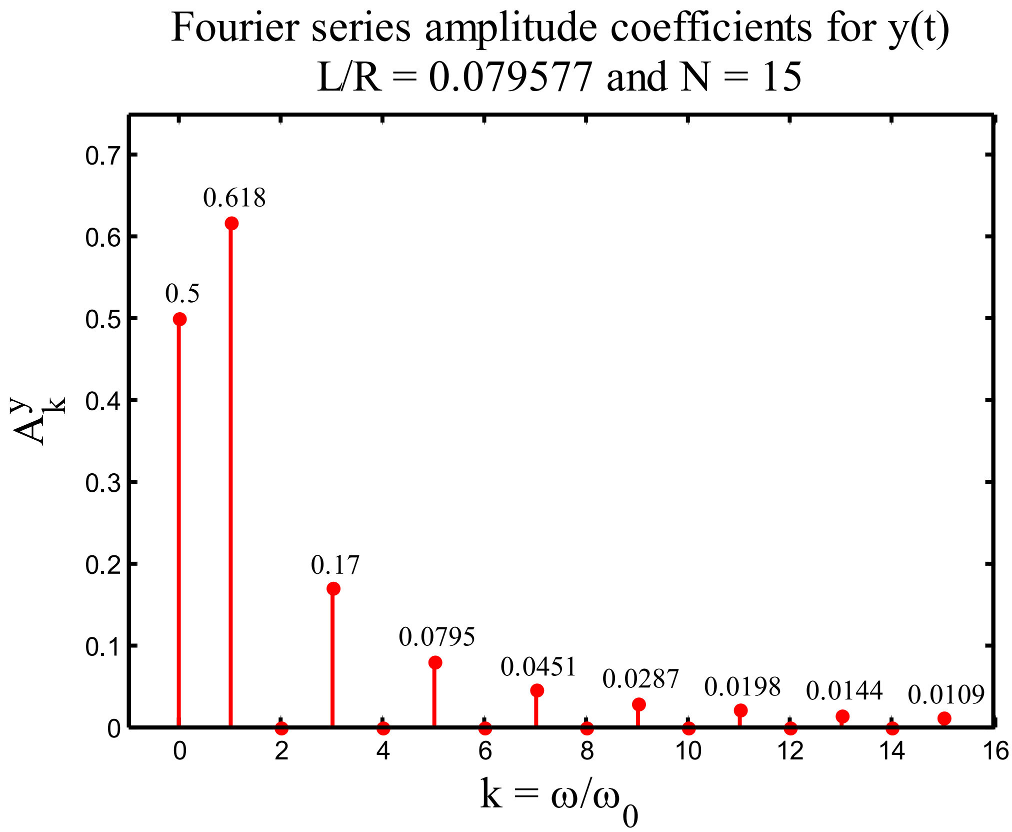
```
% chap5_rect_train_trig_FS.m
% Chapter 5 frequency response example-
% Calculate frequency response of a series LR circuit
% to a rectangular pulse train input x(t) using trig. FS.
clear;clc;close all;
T = 2; w0 = pi; % Fundamental period and frequency
t = -1.5*T:T/100:1.5*T;  % time range
LR = 1/4/pi; % ratio L/R for series LR circuit
N = 15; % Number of harmonics (excluding DC term)
%
a0 = 0.5; H0 = 1;% Input DC coeff. & DC freq. resp.
xN = a0*ones(1,length(t));     % Input dc component
yN = a0*H0*ones(1,length(t)); % Output dc component
for k=1:N,
  akx(k) = (2/k/w0)*sin(0.5*k*w0); % Input FS coeff.
  Akx(k) = abs(akx(k));          % Input FS ampl. coeff.
  Thetakx(k) = angle(akx(k));   % Input FS angle coeff.
  Hwk(k) = 1./(1+j*k*w0*LR);    % System frequency response
  Aky(k) = Akx(k)*abs(Hwk(k)); % Output FS ampl. coeff.
  Thetaky(k) = Thetakx(k) + angle(Hwk(k)); % Output FS angle coeff.
  xN = xN + Akx(k)*cos(k*w0*t+Thetakx(k)); % Input FS
  yN = yN + Aky(k)*cos(k*w0*t+Thetaky(k)); % Output FS
end
% tack on DC terms for line spectra plots
Akx =[a0,Akx]; Thetakx =[0,Thetakx]; Hwk =[H0,Hwk];Aky =[a0*H0,Aky];
% Plot input amplitude spectra Akx
k = 0:1:N; wk = k*w0;  % indices and discrete freqs for FS
stem(k,Akx,'r.'),axis([-1 N+1 0 0.75]),
ylabel(['A_k^x'],'fontsize',16,'fontname','times')
title('Fourier series amplitude coefficients for x(t)',...
    'fontsize',16,'fontname','times');
xlabel('k = \omega/\omega_0','fontsize',16,...
    'fontname','times')
for n=1:N+1,  % Label stems
    if(Akx(n)>0.0009),
        text(k(n),Akx(n)+0.05,[' ' num2str(Akx(n),3)],...
      'HorizontalAlignment','center','VerticalAlignment','top')
    end
end
% Plot |H(wk)| magnitude of frequency response at wk
figure,stem(k,abs(Hwk),'r.'),axis([-1 N+1 0 1.1]),
ylabel(['|H(\omega_k )|'],'fontsize',16,'fontname','times')
title({'Frequency response at Fourier series harmonics for x(t)';...
    ['L/R = ',num2str(LR),' and N = ',num2str(N)]},...
    'fontsize',16,'fontname','times');
xlabel('k = \omega/\omega_0','fontsize',16,...
    'fontname','times')
for n=1:N+1,  % Label stems
```

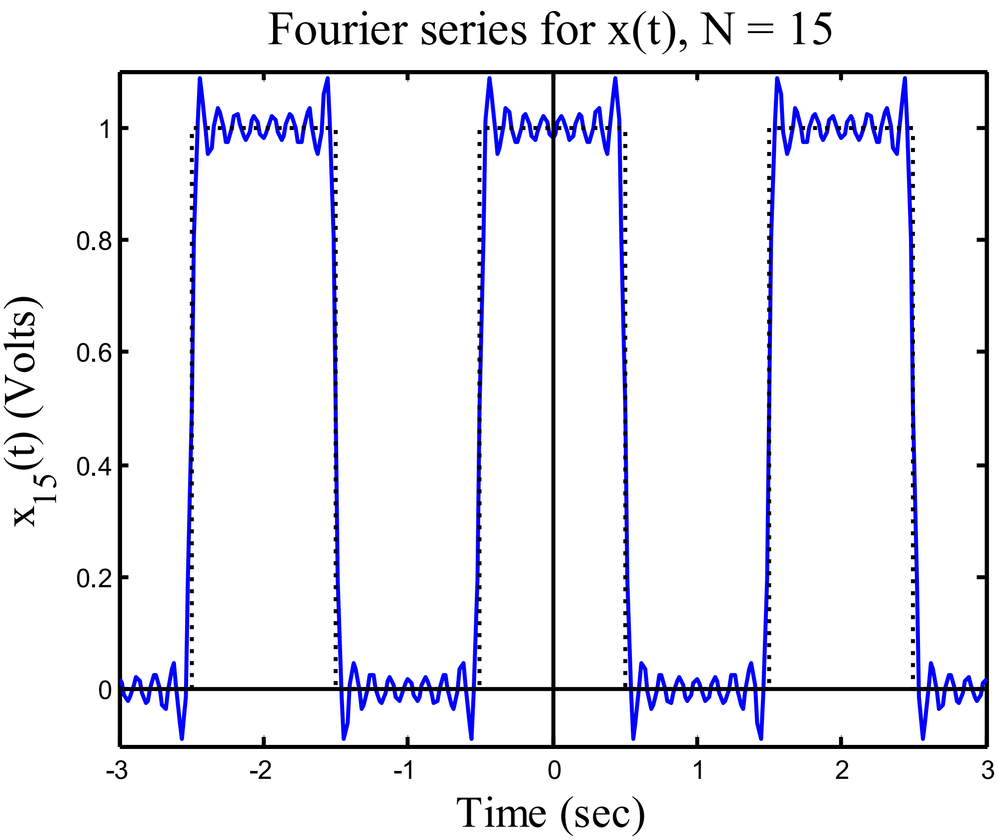
```
    if(abs(Hwk(n))>0.0009),
        text(k(n),abs(Hwk(n))+0.07,[' ' num2str(abs(Hwk(n)),3)],...
      'HorizontalAlignment','center','VerticalAlignment','top')
    end
end
% Plot output amplitude spectra Aky
figure,stem(k,Aky,'r.'),axis([-1 N+1 0 0.75]),
ylabel(['A_k^y'],'fontsize',16,'fontname','times')
title({'Fourier series amplitude coefficients for y(t)';...
    ['L/R = ',num2str(LR),' and N = ',num2str(N)]},...
    'fontsize',16,'fontname','times');
xlabel('k = \omega/\omega_0','fontsize',16,...
    'fontname','times')
for n=1:N+1,  % Label stems
    if(Aky(n)>0.0009),
        text(k(n),Aky(n)+0.05,[' ' num2str(Aky(n),3)],...
      'HorizontalAlignment','center','VerticalAlignment','top')
    end
end
% Plot x(t) and cosine/w phase trig. Fourier series for x(t)
xexact = [0 0 1 1 0 0 1 1 0 0 1 1 0 0];
texact = [-3 -2.5 -2.5 -1.5 -1.5 -0.5 -0.5 0.5 0.5 1.5 1.5 2.5 2.5 3];
figure,plot(t,xN,'b-',texact,xexact,'k:',[-3 3],[0,0],'k-',...
    [0 0],[-0.1,1.1],'k-'),
axis([-3 3 -0.1 1.1]),
title(['Fourier series for x(t), N = ',num2str(N)],...
    'fontsize',18,'fontname','times');
xlabel('Time (sec)','fontsize',16,'fontname','times');
ylabel(['x_{',num2str(N),'}(t) (Volts)'],'fontsize',16,...
    'fontname','times');
% Plot x(t) & cosine/w phase trig. Fourier series for y(t)
figure, plot(t,yN,'r-',texact,xexact,'k:',[-3 3],[0,0],'k-',...
    [0 0],[-0.1,1.1],'k-'),
axis([-3 3 -0.1 1.1]),
title({'Fourier series for y(t)';...
    ['L/R = ',num2str(LR),' and N = ',num2str(N)]},...
    'fontsize',16,'fontname','times');
xlabel('Time (sec)','fontsize',16,'fontname','times');
ylabel(['y_{',num2str(N),'}(t) (Volts)'],'fontsize',16,...
    'fontname','times')
set(findobj('type','line'),'linewidth',1.5)
set(findobj('type','text'),'fontname','times')
set(findobj('type','axes'),'linewidth',2)
set(findobj('type','line'),'markersize',14)
```

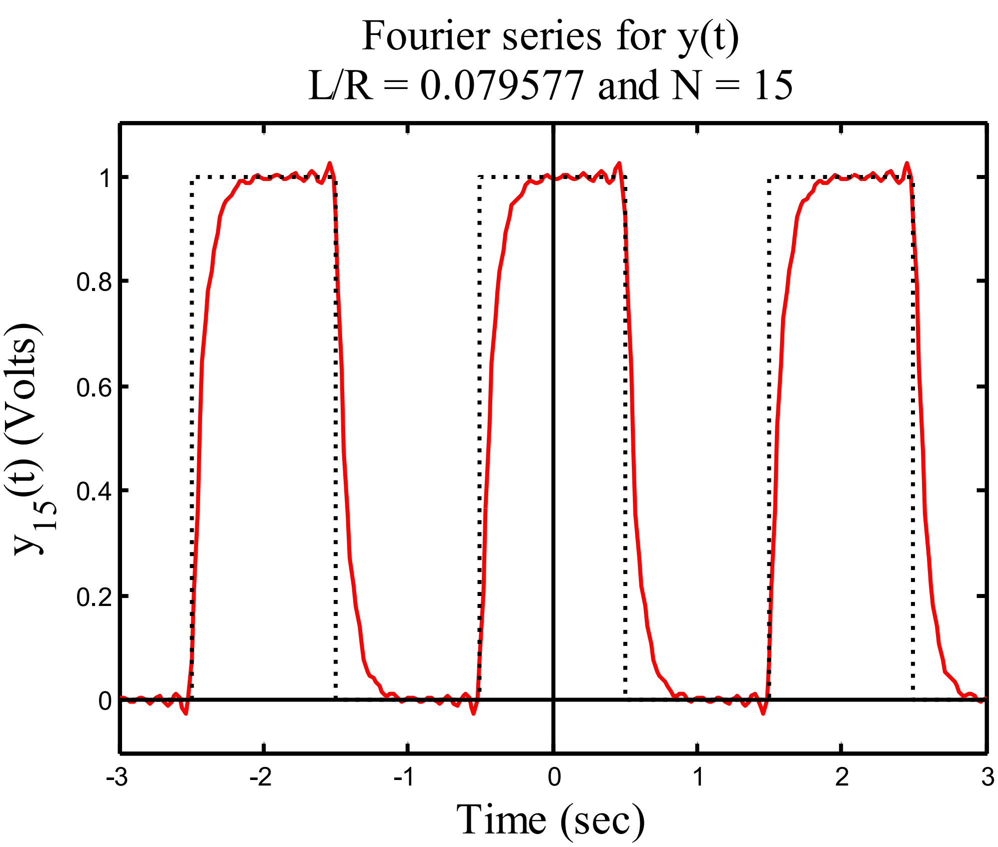
L/R = 1/(4π).


Fourier series amplitude coefficients for x(t)
0.637
0.5
0.212
0.127
0.0909
0.0707
0.0579
0.049
0.0424
A^x_k
k = ω/ω$_0$

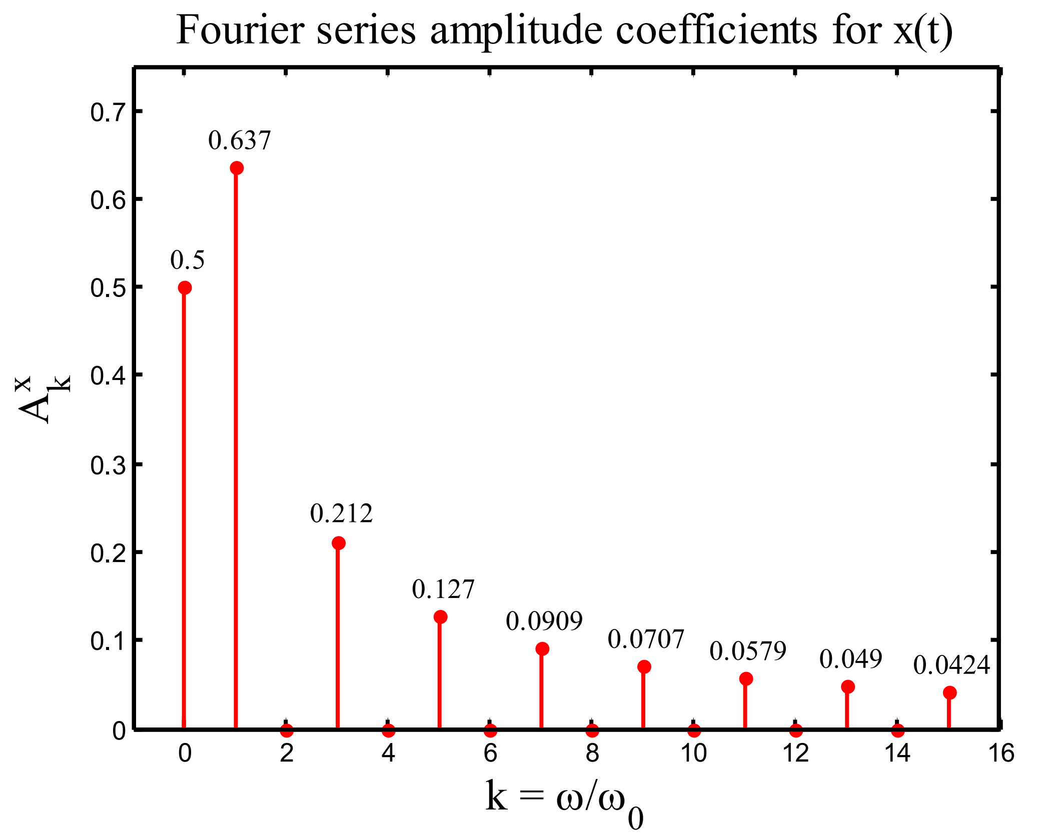

Frequency response at Fourier series harmonics for x(t)
L/R = 0.079577 and N = 15
1
0.97
0.894
0.8
0.707
0.625
0.555
0.496
0.447
0.406
0.371
0.342
0.316
0.294
0.275
0.258
|H(ω$_k$)|
k = ω/ω$_0$

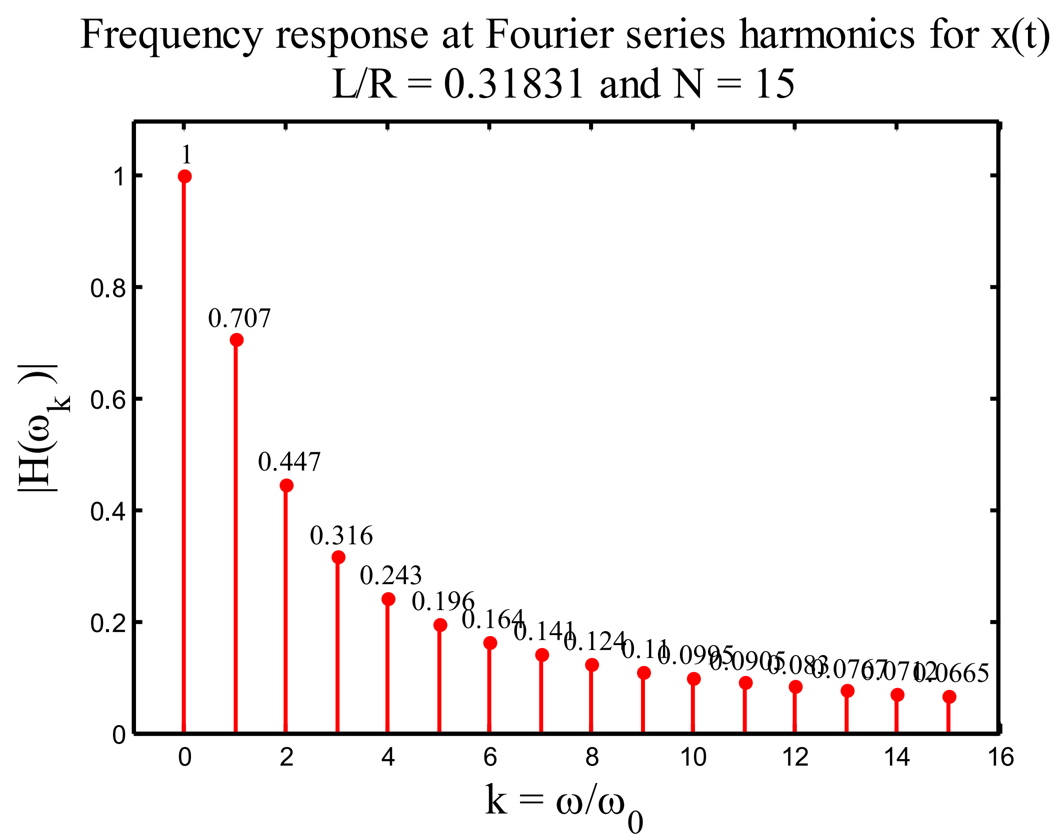

Fourier series amplitude coefficients for y(t)
L/R = 0.079577 and N = 15
A_k^y
$k = \omega/\omega_0$
0.5
0.618
0.17
0.0795
0.0451
0.0287
0.0198
0.0144
0.0109
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
2
4
6
8
10
12
14
16

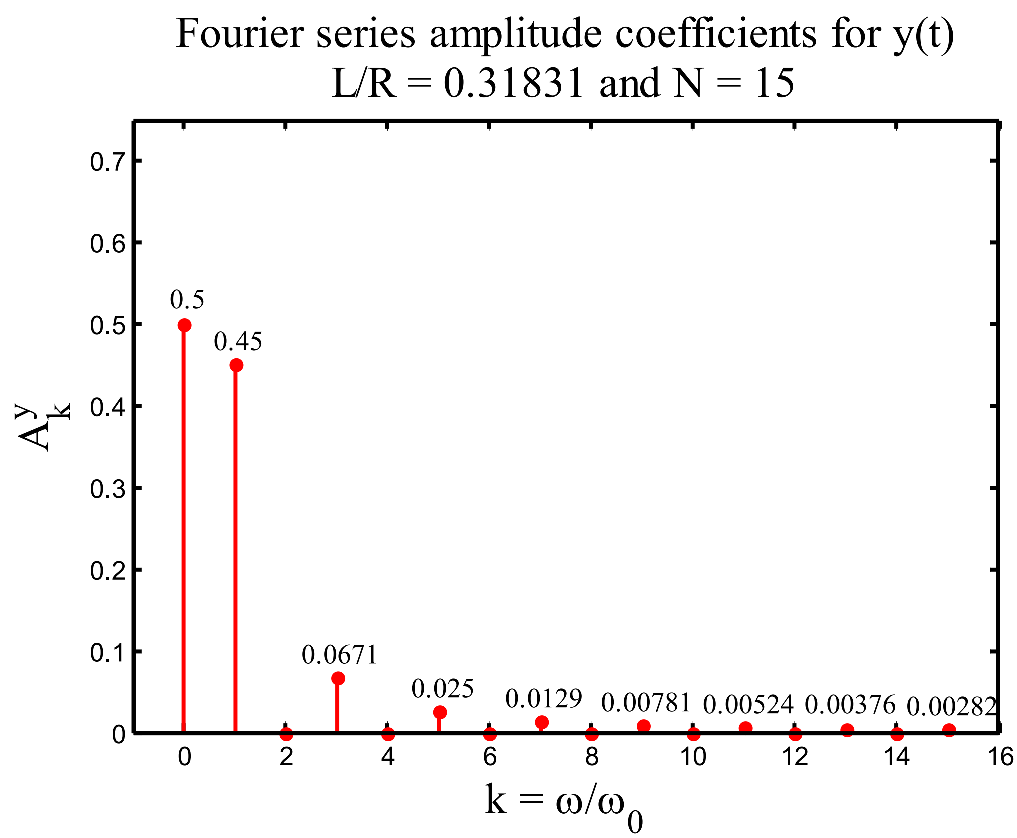

Fourier series for x(t), N = 15
$x_{15}(t)$ (Volts)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3

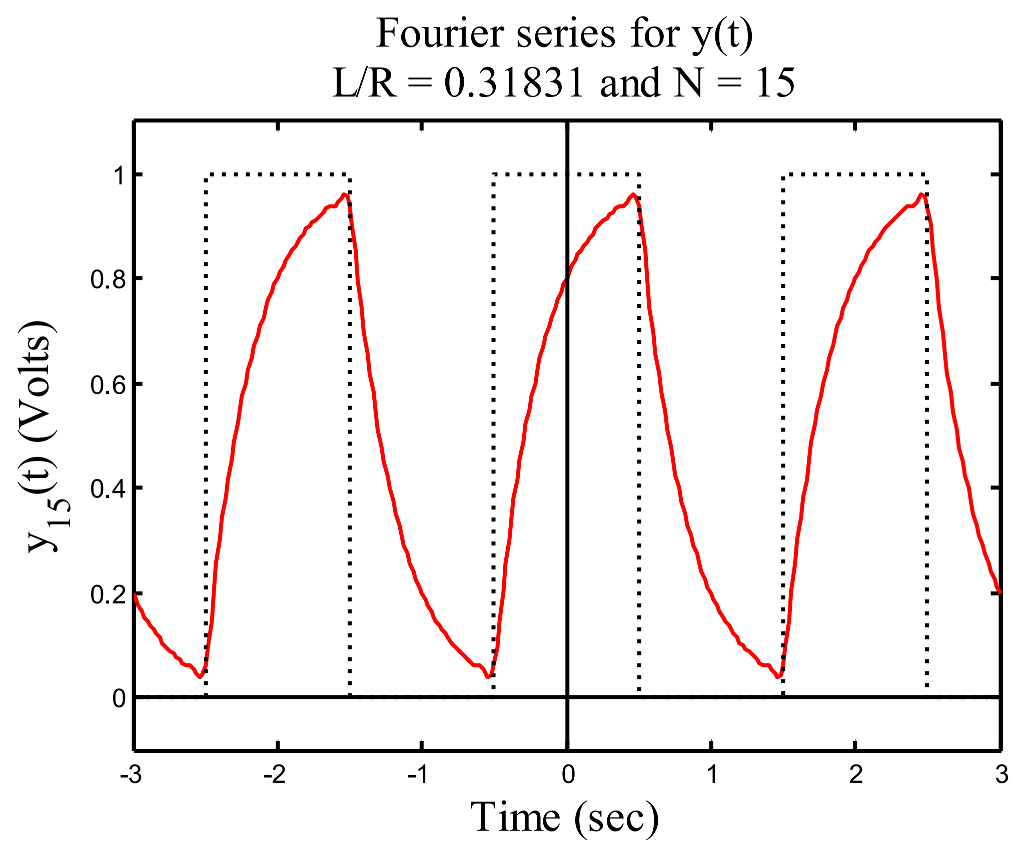

Fourier series for y(t)
L/R = 0.079577 and N = 15
$y_{15}(t)$ (Volts)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3

Next L/R = 1/π.

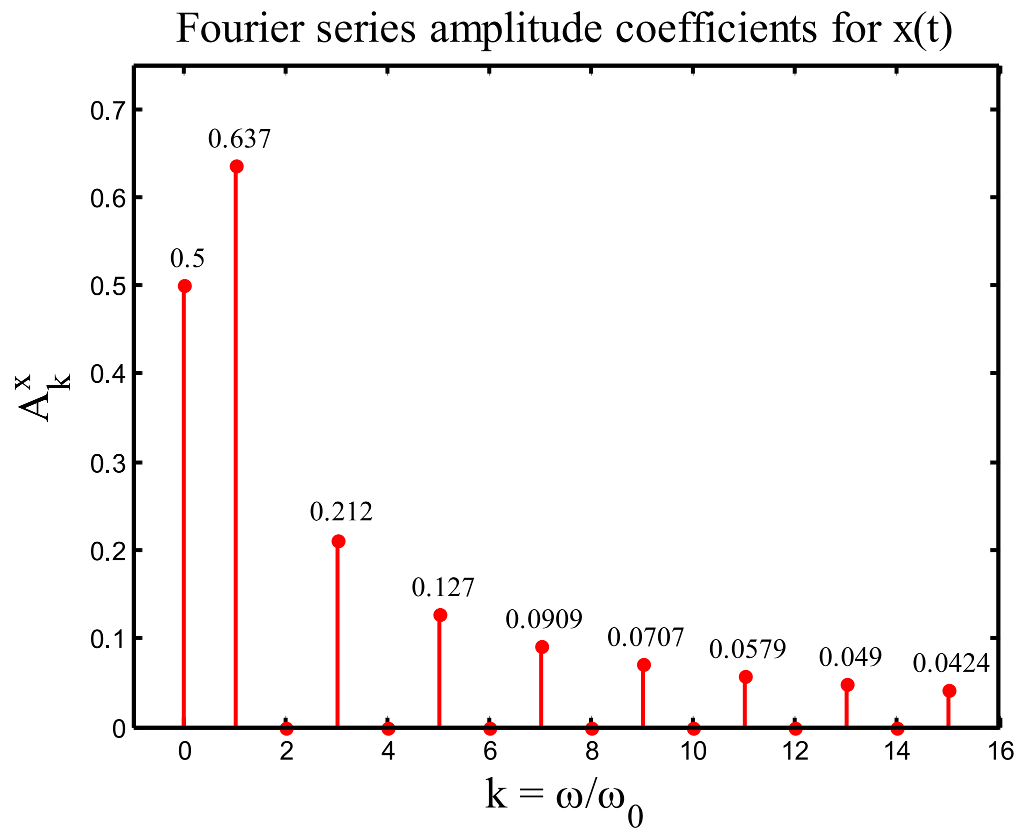

Fourier series amplitude coefficients for x(t)
A_k^x
$k = \omega/\omega_0$
0.5
0.637
0.212
0.127
0.0909
0.0707
0.0579
0.049
0.0424

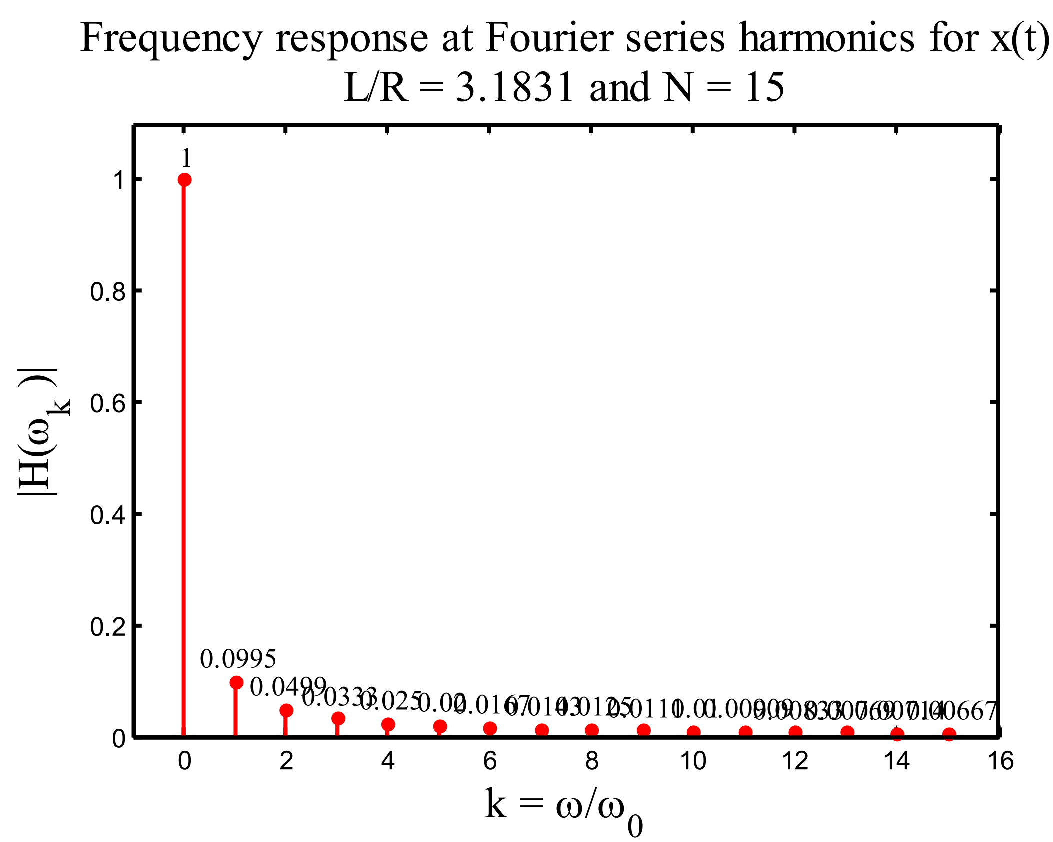

Frequency response at Fourier series harmonics for x(t)
L/R = 0.31831 and N = 15
$|H(\omega_k)|$
$k = \omega/\omega_0$
1
0.707
0.447
0.316
0.243
0.196
0.164
0.141
0.124
0.0665

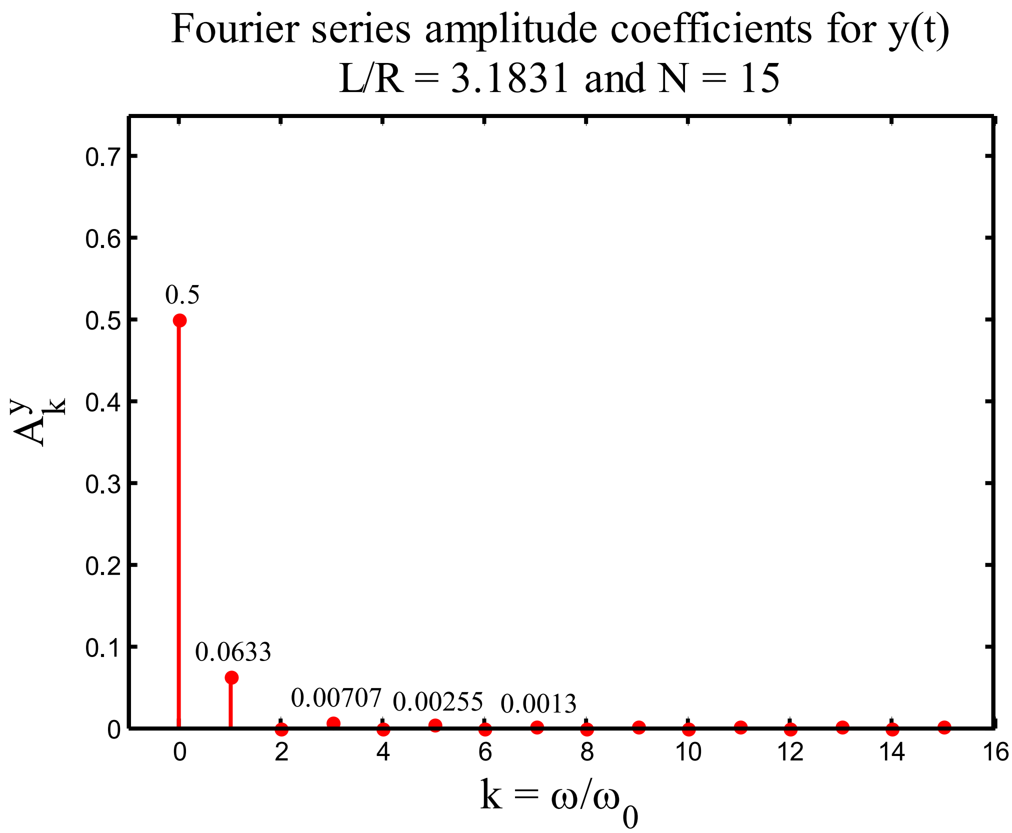

Fourier series amplitude coefficients for y(t)
L/R = 0.31831 and N = 15
A_k^y
0.5
0.45
0.0671
0.025
0.0129
0.00781
0.00524
0.00376
0.00282
$k = \omega/\omega_0$

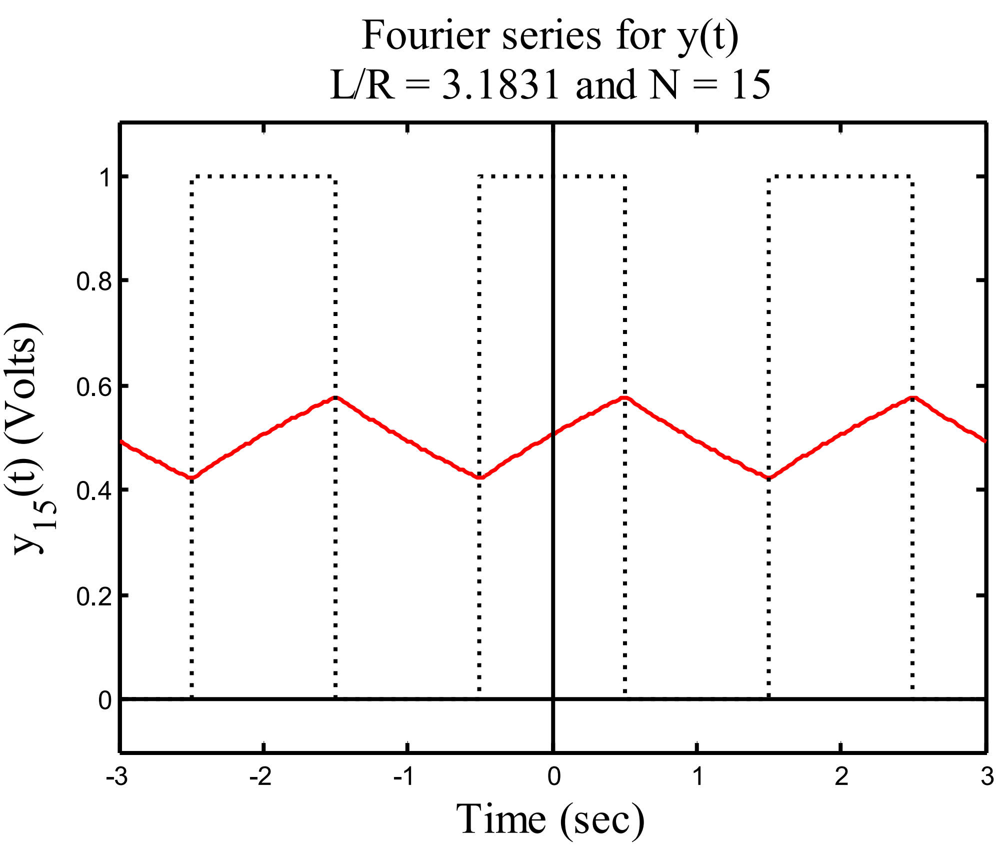

Fourier series for y(t)
L/R = 0.31831 and N = 15
$y_{15}(t)$ (Volts)
Time (sec)

Next L/R = 10/π.


Fourier series amplitude coefficients for x(t)
0.5
0.637
0.212
0.127
0.0909
0.0707
0.0579
0.049
0.0424
A_k^x
$k = \omega/\omega_0$


Frequency response at Fourier series harmonics for x(t)
L/R = 3.1831 and N = 15
1
0.0995
$|H(\omega_k)|$
$k = \omega/\omega_0$


Fourier series amplitude coefficients for y(t)
L/R = 3.1831 and N = 15
A^y_k
0.5
0.0633
0.00707
0.00255
0.0013
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
k = ω/ω$_0$


Fourier series for y(t)
L/R = 3.1831 and N = 15
$y_{15}(t)$ (Volts)
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
Time (sec)

Alternately, we could do this using the complex exponential Fourier series (actually easier to implement in the MATLAB code).

Here, since $x(t)$ is even, the complex exponential Fourier series coefficients are

$$c_0^x = a_0 = 0.5 \text{ for } k = 0 \text{ (DC term)}$$

and

$$c_k^x = \frac{a_k}{2} = \frac{1}{k\pi}\sin(0.5k\pi) \text{ for } k = \pm 1, \pm 2, \pm 3, \ldots \text{ (harmonics).}$$

The complex exponential Fourier series for $x(t)$ is then

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^x \, e^{jk\pi t} .$$

The complex exponential Fourier series coefficients for the output $y(t)$ are simply obtained by multiplying the $x(t)$ coefficients by the frequency response $H(\omega)$ of the series LR circuit at the frequencies corresponding to the harmonics, i.e.,

$$H(\omega = \omega_k = k\pi) = \frac{1}{1 + jk\pi\left(L/R\right)} .$$

This yields

$$c_0^y = c_0^x \, H(0) = a_0(1) = 0.5 \text{ for } k = 0 \text{ (DC term)}$$

and

$$c_k^y = c_k^x \, H(k\pi) = \frac{1}{k\pi}\sin(0.5k\pi)\left(\frac{1}{1 + jk\pi\left(L/R\right)}\right)$$

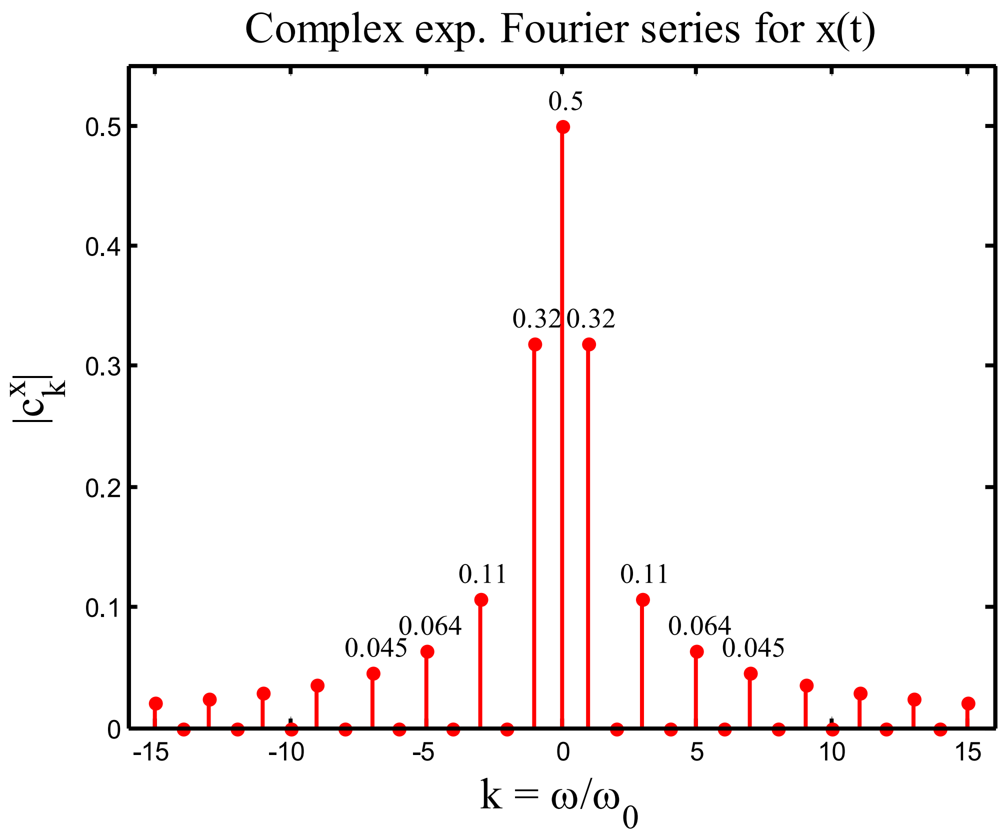
for $k = \pm 1, \pm 2, \pm 3, \ldots$ (harmonics). The complex exponential Fourier series for y(t) is then

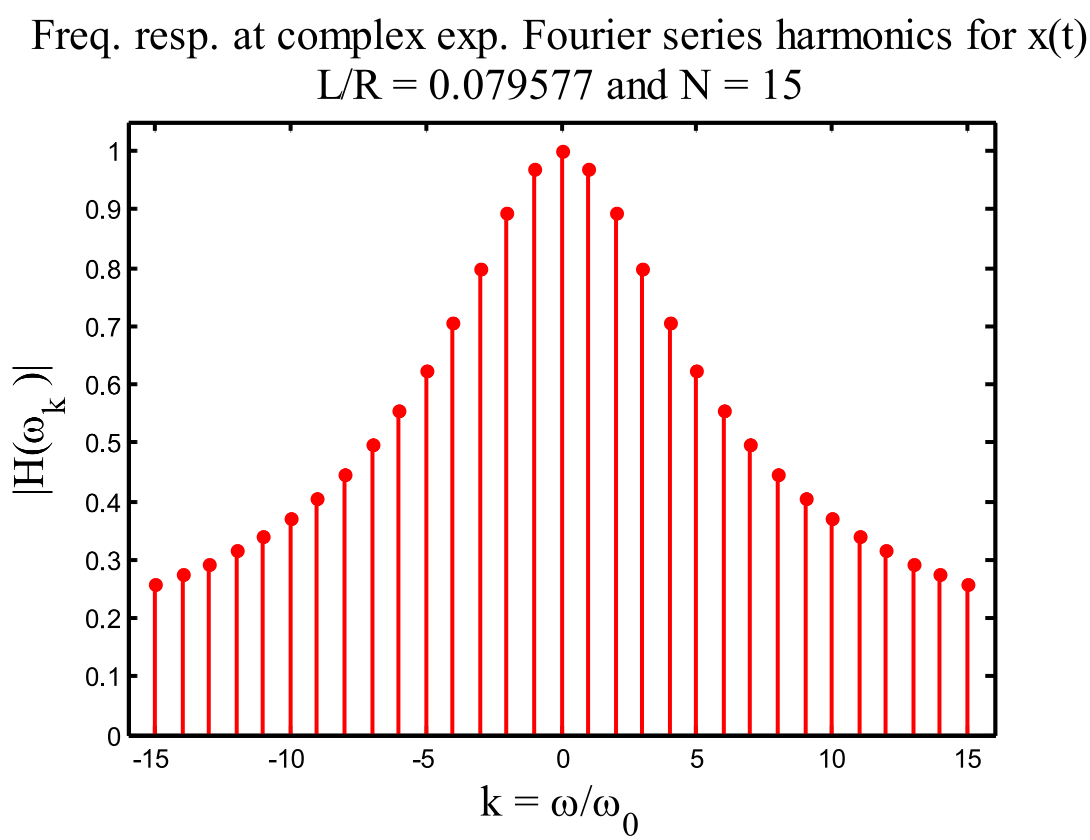
$$y(t) = \sum_{k=-\infty}^{\infty} c_k^y \, e^{jk\pi t} .$$

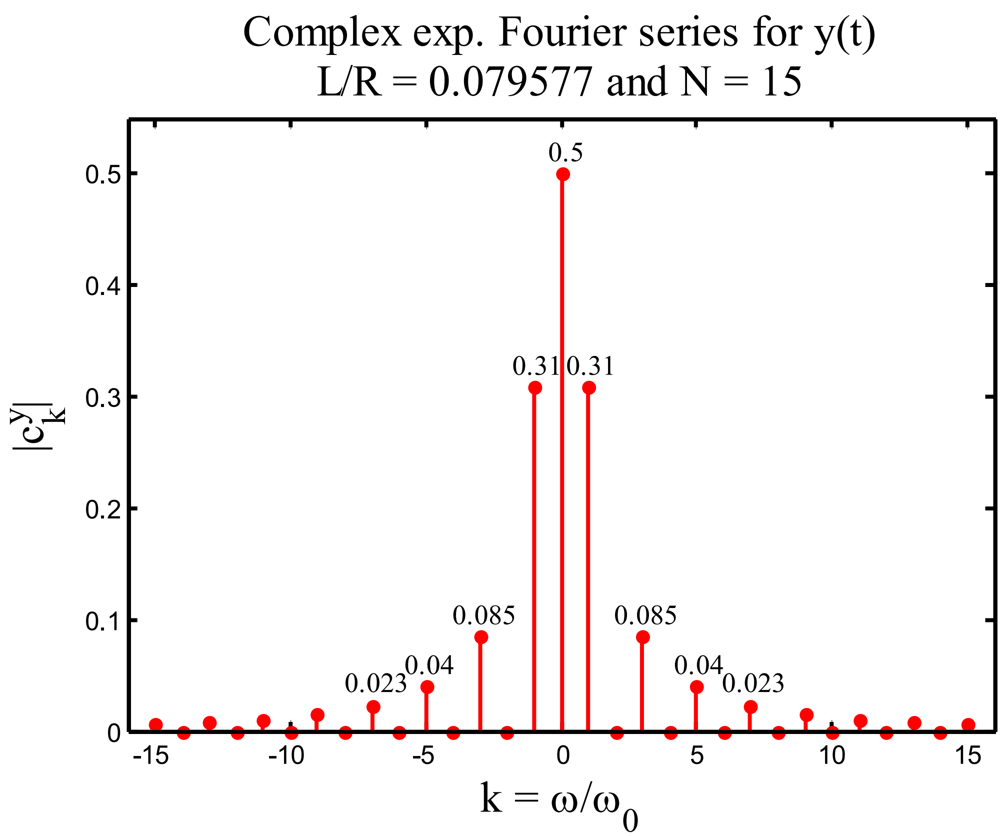
Once again, in the MATLAB file, we go to the N^{th} harmonic instead of ∞ in the summation.

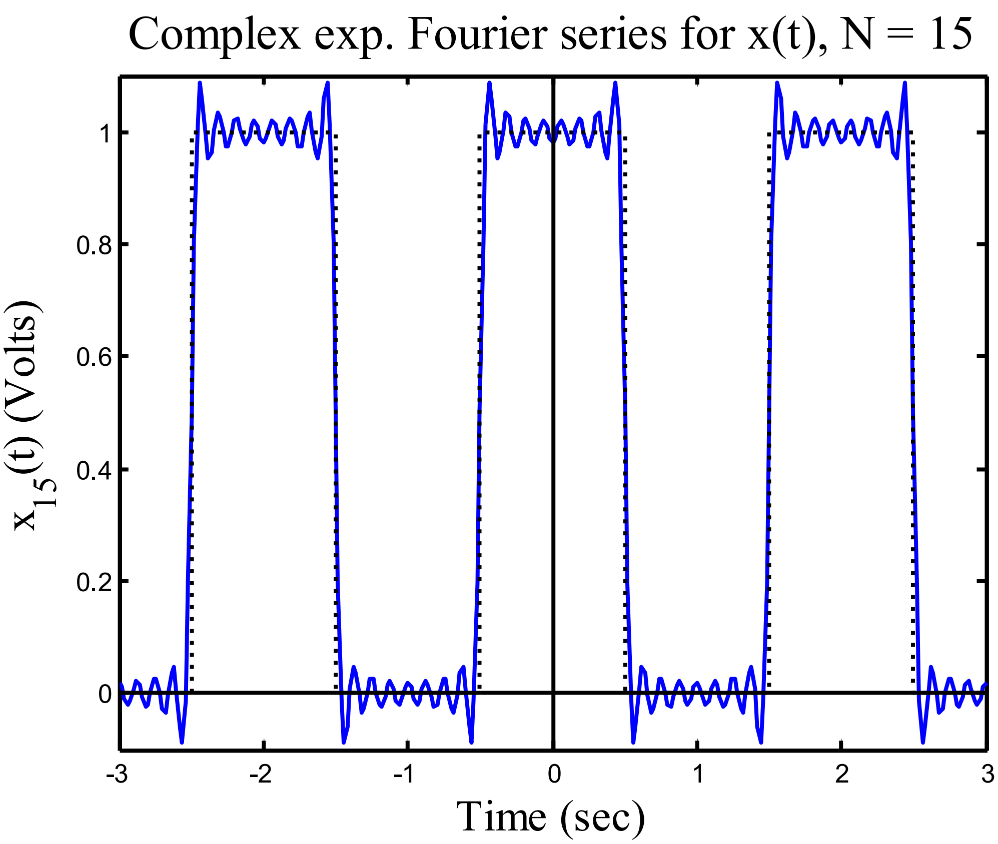
```
% chap5_rect_train_complex_exp_FS.m
% Chapter 5 frequency response example-
% Calculate frequency response of a series LR circuit
% to a rectangular pulse train input x(t) using complex
% exponential Fourier series.
clear;clc;close all;
T = 2; w0 = pi; % Fundamental period and frequency
t = -1.5*T:T/100:1.5*T;  % time range
LR = 1/4/pi; % ratio L/R for series LR circuit
N = 15; % Number of harmonics (excluding DC term)
%
c0 = 0.5; % Input DC coeff
xN = zeros(1,length(t)); % initialize input vector
yN = zeros(1,length(t)); % initialize output vector
for k1 = 1:1:2*N+1,
    ktmp = k1-(N+1); % Count from -N to +N
% Calculate complex exp. coeff. for input and output
    if(ktmp == 0),
        ckx(k1) = c0;    % DC coeff.
        Hwk(k1) = 1.0;   % System DC freq. response
        cky(k1) = c0*Hwk(k1);
    else
        ckx(k1) = (1/ktmp/w0).*sin(0.5*ktmp*w0);
        Hwk(k1) = 1./(1+j*ktmp*w0*LR);   % System freq. response
        cky(k1) = ckx(k1)*Hwk(k1);
    end
    xN = xN + ckx(k1)*exp(1j*ktmp*w0*t);
    yN = yN + cky(k1)*exp(1j*ktmp*w0*t);
end
% Plot input amplitude spectra ckx
k = -N:1:N; wk = k*w0;  % indices and discrete freqs for FS
stem(k,abs(ckx),'r.'),axis([-N-1 N+1 0 0.55]),
ylabel(['|c_k^x|'],'fontsize',16,'fontname','times')
title('Complex exp. Fourier series for x(t)',...
    'fontsize',16,'fontname','times');
xlabel('k = \omega/\omega_0','fontsize',16,...
    'fontname','times')
for n=1:2*N+1,  % Label stems
    if(abs(ckx(n))>0.04),
        text(k(n),abs(ckx(n))+0.035,[' ' num2str(abs(ckx(n)),2)],...
      'HorizontalAlignment','center','VerticalAlignment','top')
    end
end
% Plot |H(wk)| magnitude of frequency response at wk
figure,stem(k,abs(Hwk),'r.'),axis([-N-1 N+1 0 1.05]),
ylabel(['|H(\omega_k )|'],'fontsize',16,'fontname','times')
title({'Freq. resp. at complex exp. Fourier series harmonics for
x(t)';...
```

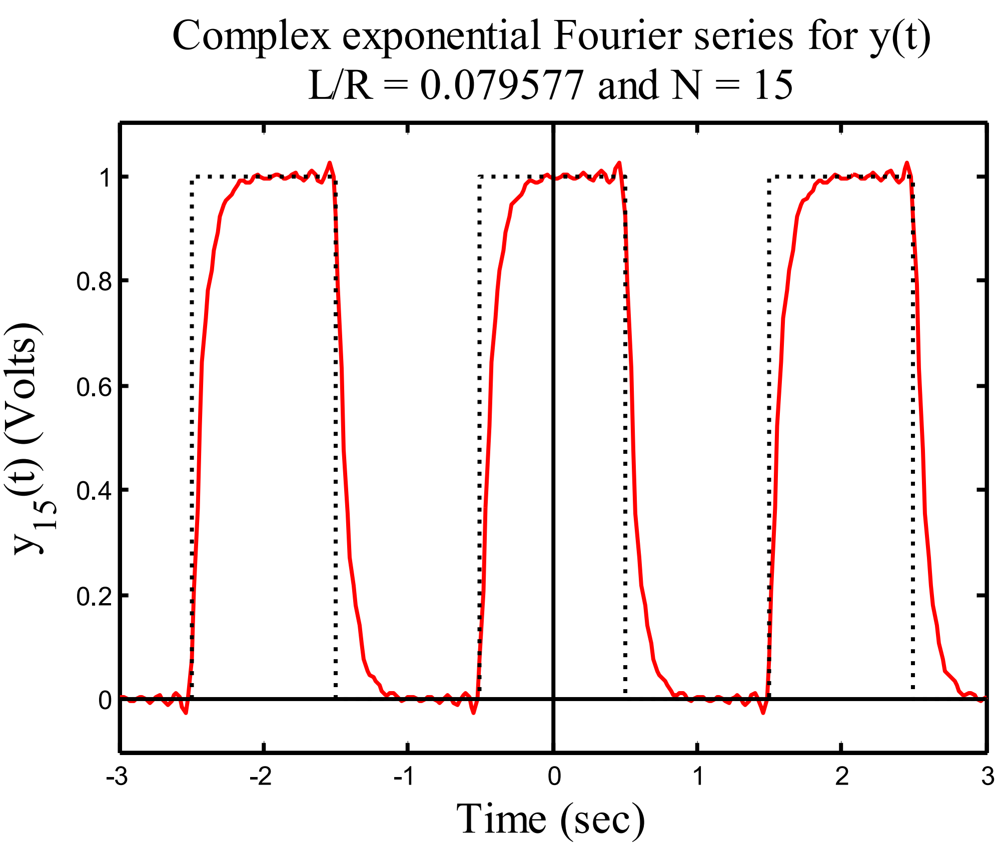
```
    ['L/R = ',num2str(LR),' and N = ',num2str(N)]},...
    'fontsize',16,'fontname','times');
xlabel('k = \omega/\omega_0','fontsize',16,...
    'fontname','times')
% Plot output amplitude spectra cky
figure,stem(k,abs(cky),'r.'),axis([-N-1 N+1 0 0.55]),
ylabel(['|c_k^y|'],'fontsize',16,'fontname','times')
title({'Complex exp. Fourier series for y(t)',...
    ['L/R = ',num2str(LR),' and N = ',num2str(N)]},...
    'fontsize',16,'fontname','times');
xlabel('k = \omega/\omega_0','fontsize',16,...
    'fontname','times')
for n=1:2*N+1,  % Label stems
    if(abs(cky(n))>0.02),
        text(k(n),abs(cky(n))+0.035,[' ' num2str(abs(cky(n)),2)],...
      'HorizontalAlignment','center','VerticalAlignment','top')
    end
end
% Plot x(t) and complex exp. Fourier series for x(t)
xexact = [0 0 1 1 0 0 1 1 0 0 1 1 0 0];
texact = [-3 -2.5 -2.5 -1.5 -1.5 -0.5 -0.5 0.5 0.5 1.5 1.5 2.5 2.5 3];
figure,plot(t,xN,'b-',texact,xexact,'k:',[-3 3],[0,0],'k-',...
    [0 0],[-0.1,1.1],'k-'),
axis([-3 3 -0.1 1.1]),
title(['Complex exp. Fourier series for x(t), N = ',num2str(N)],...
    'fontsize',18,'fontname','times');
xlabel('Time (sec)','fontsize',16,'fontname','times');
ylabel(['x_{',num2str(N),'}(t) (Volts)'],'fontsize',16,...
    'fontname','times');
% Plot x(t) & complex exp. Fourier series for y(t)
figure, plot(t,yN,'r-',texact,xexact,'k:',[-3 3],[0,0],'k-',...
    [0 0],[-0.1,1.1],'k-'),
axis([-3 3 -0.1 1.1]),
title({'Complex exponential Fourier series for y(t)';...
    ['L/R = ',num2str(LR),' and N = ',num2str(N)]},...
    'fontsize',16,'fontname','times');
xlabel('Time (sec)','fontsize',16,'fontname','times');
ylabel(['y_{',num2str(N),'}(t) (Volts)'],'fontsize',16,...
    'fontname','times')
set(findobj('type','line'),'linewidth',1.5)
set(findobj('type','text'),'fontname','times')
set(findobj('type','axes'),'linewidth',2)
set(findobj('type','line'),'markersize',16)
```


Complex exp. Fourier series for x(t)
0.5
0.32
0.32
0.11
0.11
0.064
0.064
0.045
0.045
$|c^x_k|$
$k = \omega/\omega_0$


Freq. resp. at complex exp. Fourier series harmonics for x(t)
L/R = 0.079577 and N = 15
$|H(\omega_k)|$
$k = \omega/\omega_0$


Complex exp. Fourier series for y(t)
L/R = 0.079577 and N = 15
0.5
0.31
0.31
0.085
0.085
0.04
0.04
0.023
0.023
0.5
0.4
0.3
0.2
0.1
0
-15
-10
-5
0
5
10
15
k = ω/ω0


Complex exp. Fourier series for x(t), N = 15
$x_{15}(t)$ (Volts)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3


Complex exponential Fourier series for y(t)
L/R = 0.079577 and N = 15
$y_{15}(t)$ (Volts)
Time (sec)
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3