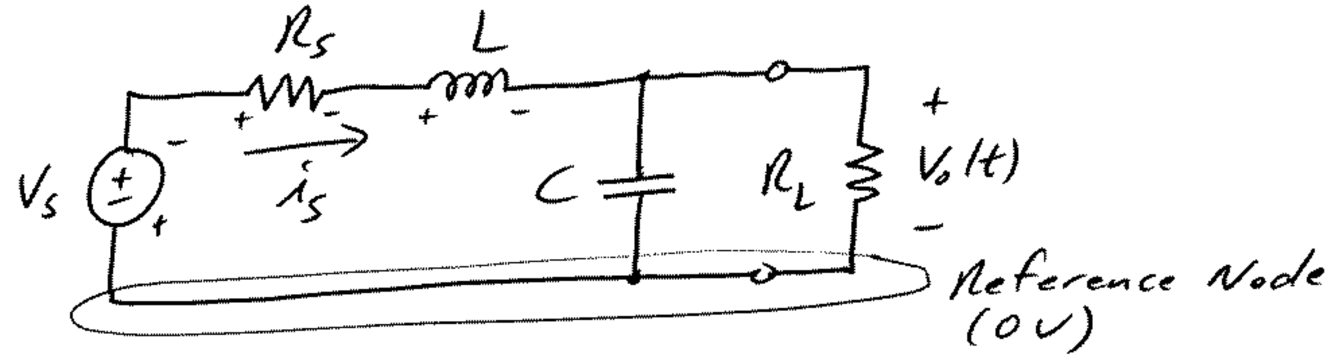
Obtain the differential equation for the output voltage $v_O(t)$ of the power supply circuit shown. Then, determine $v_O(t)$ when $v_S = 12\,u(t)$ V, $R_S = 8\,\Omega$, $L = 10$ mH, $C = 47\,\mu$F, and $R_L = 8\,\Omega$ for all time $t \geq 0$.



① Apply KVL to outer loop

$$-V_S + i_S R_S + L\frac{di_S}{dt} + V_o(t) = 0$$

② Apply KCL to top right node

$$i_S = C\frac{dV_o}{dt} + \frac{V_o}{R_L}$$

③ Substitute i_S into first equation

$$\left(C\frac{dV_o}{dt} + \frac{V_o}{R_L}\right)R_S + L\frac{d}{dt}\left(C\frac{dV_o}{dt} + \frac{V_o}{R_L}\right) + V_o = V_S$$

$$R_S C\frac{dV_o}{dt} + \frac{R_S}{R_L}V_o + LC\frac{d^2V_o}{dt^2} + \frac{L}{R_L}\frac{dV_o}{dt} + V_o = V_S$$

do a little algebra

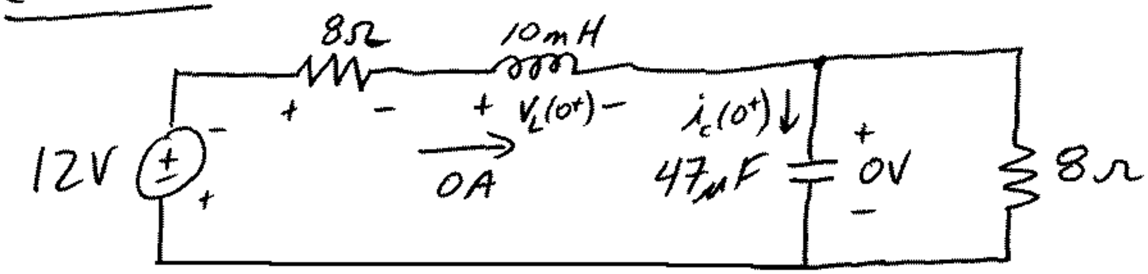
$$\underline{\underline{\frac{d^2V_o}{dt^2} + \frac{1}{LC}\left(R_S C + \frac{L}{R_L}\right)\frac{dV_o}{dt} + \frac{1}{LC}\left(\frac{R_S}{R_L} + 1\right)V_o = \frac{V_S}{LC}}}$$

Find initial & final conditions

@ $t=0^-$, $V_S = 0 \Rightarrow i_S(0^-) = i_L(0^-) = \underline{0 = i_L(0)}$

$$\Rightarrow V_o(0^-) = V_C(0^-) = \underline{0 = V_o(0)}$$

@ $t=0^+$ Circuit is in transition



Apply KVL $\quad -12 + 0(8\Omega) + V_L(0^+) + 0 = 0$

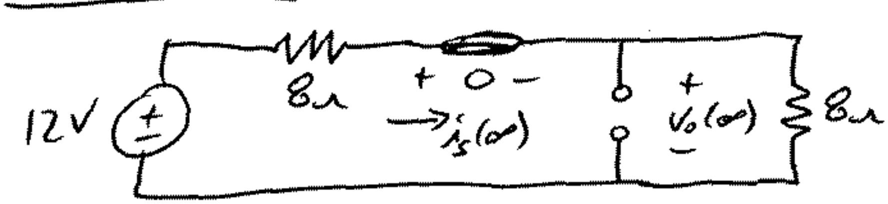
$$V_L(0^+) = 10\times10^{-3} \left.\frac{di_L}{dt}\right|_{t=0^+} = 12$$

$$\underline{\left.\frac{di_L}{dt}\right|_{t=0^+} = 1200\ A/s}$$

Apply KCL top right node $\quad 0 + i_C(0^+) + \frac{0V}{8\Omega} = 0$

$$\left(47\times10^{-6}\right)\left.\frac{dV_o}{dt}\right|_{t=0^+} = i_C(0^+) = 0 \Rightarrow \underline{\left.\frac{dV_o}{dt}\right|_{t=0^+} = 0}$$

@ $t \to \infty$ Circuit is at steady-state ($V_L = 0$, $i_C = 0$)



$i_S(\infty) = \frac{12}{8+8} = \underline{0.75A}$

$\underline{V_o(\infty) = 12\left(\frac{8}{8+8}\right) = 6V}$

Determine characteristic equation for natural response ($V_S = 0$)

$$s^2 + \frac{1}{LC}\left(R_S C + \frac{L}{R_L}\right)s + \frac{1}{LC}\left(\frac{R_S}{R_L} + 1\right) = 0$$

$$s^2 + \frac{1}{10\times10^{-3}(47\times10^{-6})}\left(8(47\times10^{-6}) + \frac{10\times10^{-3}}{8}\right)s$$

$$+ \frac{1}{10\times10^{-3}(47\times10^{-6})}\left(\frac{8}{8} + 1\right) = 0$$

$$s^2 + 3459.574468\ s + 4{,}255{,}319.149 = 0$$ TI-68

$$s_1 = -1729.787 + j\,1123.9018\ s^{-1}$$

$$s_2 = -1729.787 - j\,1123.9018\ s^{-1}$$

$\Downarrow$ underdamped

Natural Response

$$V_{o,n}(t) = e^{-1729.8t}\left[A_1 \cos(1123.9t) + A_2 \sin(1123.9t)\right]$$

Forced Response

$$V_{o,f}(t) = V_o(\infty) = 6\,V$$

Total Response

$$V_o(t) = V_{o,n}(t) + V_{o,f}(t)$$ ← General solution

$$V_o(t) = e^{-1729.8t}\left[A_1 \cos(1123.9t) + A_2 \sin(1123.9t)\right] + 6\,V$$

Find particular solution (i.e., apply initial conditions)

$$V_o(0) = 0 = e^0\left[A_1 \cos(0) + A_2 \sin(0)\right] + 6\,V$$

$$\Rightarrow \underline{A_1 = -6\,V}$$

$$V_o(t) = e^{-1729.8t}\left[-6\cos(1123.9t) + A_2 \sin(1123.9t)\right] + 6\,V$$

$$\frac{dV_o}{dt} = -1729.8\,e^{-1729.8t}\left[-6\cos(1123.9t) + A_2\sin(1123.9t)\right] + e^{-1729.8t}\left[+6(1123.9)\sin(1123.9t) + A_2(1123.9)\cos(1123.9t)\right]$$

$$\left.\frac{dV_o}{dt}\right|_{t=0} = 0 = -1729.8(1)\left[-6\overset{1}{\cos}(0) + A_2\overset{0}{\sin}(0)\right] + (1)\left[6(1123.9)\overset{0}{\sin}(0) + A_2(1123.9)\overset{1}{\cos}(0)\right]$$

$$\hookrightarrow A_2 = \frac{-6(1729.8)}{1123.9} = \underline{-9.234545}$$

$$\underline{\underline{V_o(t) = e^{-1729.8t}\left[-6\cos(1123.9t) - 9.234545\sin(1123.9t)\right] + 6\,V}}$$

$$\underline{\underline{t \geq 0}}$$

Power Supply Circuit Step Response (underdamped)

$n := 0 .. 2000 \qquad t_n := n \cdot 3 \cdot 10^{-6} \quad s$

$\alpha := 1729.787 \quad np/s \qquad \omega d := 1123.9018 \quad rad/s$

$vo(t) := e^{-\alpha \cdot t} \cdot \left(-6 \cdot \cos(\omega d \cdot t) - 9.234545 \cdot \sin(\omega d \cdot t)\right) + 6 \quad V$